NATURAL PHILOSOPHY

ELEMENTARY TREATISE

ON

NATURAL PHILOSOPHY.

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BY

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TRANSLATED AND EDITED, WITH EXTENSIVE MODIFICATIONS

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IN FOUR PARTS.

PART IV.

SOUND AND LIGHT.

ILLUSTRATED BY
193 ENGRAVINGS ON WOOD, AND ONE COLORED PLATE.

TENTH EDITION.

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PREFACE TO THE TENTH EDITION OF

PART IV.

THIS Part, even in its original form, contained large portions which were re-written rather than translated, and the last two chapters had no place at all in the original French. In the sixth edition the numbering of the chapters and sections was altered, to make it consecutive with the other three parts, and additional matter was introduced under several heads. In the seventh edition two pages on Concave Diffraction Gratings were added. The tenth edition contains additions relating to beats, recent measures of the velocity of light, goniometers, convex mirrors, and nodal points of lenses.

BELFAST, *March*, 1888.

NOTE PREFIXED TO FIRST EDITION.

IN the present Part, the chapters relating to Consonance and Dissonance, Colour, the Undulatory Theory, and Polarization, are the work of the Editor; besides numerous changes and additions in other places.

The numbering of the original sections has been preserved only to the end of Chapter LX.; the two last chapters of the original having been transposed for greater convenience of treatment. With this exception, the announcements made in the "Translator's Preface," at the beginning of Part I., are applicable to the entire work.

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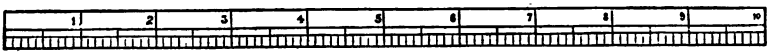
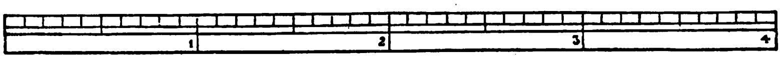
FRENCH AND ENGLISH MEASURES.

A DECIMETRE DIVIDED INTO CENTIMETRES AND MILLIMETRES.





INCHES AND TENTHS.

REDUCTION OF FRENCH TO ENGLISH MEASURES.

Length.

1 millimetre = ·03937 inch, or about $\frac{1}{25}$ inch.
1 centimetre = ·3937 inch.
1 decimetre = 3·937 inch.
1 metre = 39·37 inch = 3·281 ft. = 1·0936 yd.
1 kilometre = 1093·6 yds., or about $\frac{5}{8}$ mile.
More accurately, 1 metre = 39·370432 in. = 3·2808693 ft. = 1·09362311 yd.

Area.

1 sq. millim. = ·00155 sq. in.
1 sq. centim. = ·155 sq. in.
1 sq. decim. = 15·5 sq. in.
sq. metre = 1550 sq. in. = 10·764 sq. ft. = 1·196 sq. yd.

Volume.

1 cub. millim. = ·000061 cub. in.
1 cub. centim. = ·061025 cub. in.
1 cub. decim. = 61·0254 cub. in.
cub. metre = 61025 cub. in. = 35·3156 cub. ft. = 1·308 cub. yd.

The Litre (used for liquids) is the same as the cubic decimetre, and is equal to 1·7617 pint, or ·22021 gallon.

Mass and Weight.

1 milligramme = ·01543 grain.
1 gramme = 15·432 grain.
1 kilogramme = 15432 grains = 2·205 lbs. avoir.
More accurately, the kilogramme is 2·20462125 lbs.

Miscellaneous.

1 gramme per sq. centim. = 2·0481 lbs. per sq. ft.
1 kilogramme per sq. centim. = 14·223 lbs. per sq. in.

1 kilogrammetre = 7·2331 foot-pounds.
1 force de cheval = 75 kilogrammetres per second, or 542½ foot-pounds per second nearly, whereas 1 horse-power (English) = 550 foot-pounds per second.

REDUCTION TO C.G.S. MEASURES. (See page 48.)

[*cm.* denotes centimetre(s); *gm.* denotes gramme(s).]

Length.

1 inch = 2·54 centimetres, nearly.
1 foot = 30·48 centimetres, nearly.
1 yard = 91·44 centimetres, nearly.
1 statute mile = 160933 centimetres, nearly.
More accurately, 1 inch = 2·5399772 centimetres.

Area.

1 sq. inch = 6·45 sq. cm., nearly.
1 sq. foot = 929 sq. cm., nearly.
1 sq. yard = 8361 sq. cm., nearly.
1 sq. mile = $2·59 \times 10^{10}$ sq. cm., nearly.

Volume.

1 cub. inch = 16·39 cub. cm., nearly.
1 cub. foot = 28316 cub. cm., nearly.
1 cub. yard = 764535 cub. cm., nearly.
1 gallon = 4541 cub. cm., nearly.

Mass.

1 grain = ·0648 gramme, nearly.
1 oz. avoir. = 28·35 gramme, nearly.
1 lb. avoir. = 453·6 gramme, nearly.
1 ton = $1·016 \times 10^6$ gramme, nearly
More accurately, 1 lb. avoir. = 453·59265 gm.

Velocity.

1 mile per hour = 44·704 cm. per sec.
1 kilometre per hour = 27·7 cm. per sec.

Density.

1 lb. per cub. foot = ·016019 gm. per cub. cm.
62·4 lbs. per cub. ft. = 1 gm. per cub. cm.

FORCE (assuming $g=981$). (See p. 48.)

Weight of 1 grain $=63\cdot57$ dynes, nearly.
" 1 oz. avoir. $=2\cdot78\times10^4$ dynes, nearly.
" 1 lb. avoir. $=4\cdot45\times10^5$ dynes, nearly.
" 1 ton $=9\cdot97\times10^8$ dynes, nearly.
" 1 gramme $=981$ dynes, nearly.
" 1 kilogramme $=9\cdot81\times10^5$ dynes, nearly.

WORK (assuming $g=981$). (See p. 48.)

1 foot-pound $=1\cdot356\times10^7$ ergs, nearly.
1 kilogrammetre $=9\cdot81\times10^7$ ergs, nearly.
Work in a second by one theoretical "horse." $=7\cdot46\times10^9$ ergs, nearly.

STRESS (assuming $g=981$).

1 lb. per sq. ft. $=479$ dynes per sq. cm., nearly.
1 lb. per sq. inch $=6\cdot9\times10^4$ dynes per sq. cm., nearly.
1 kilog. per sq. cm. $=9\cdot81\times10^5$ dynes per sq. cm., nearly.
760 mm. of mercury at 0°C. $=1\cdot014\times10^6$ dynes per sq. cm., nearly.
30 inches of mercury at 0° C. $=1\cdot0163\times10^6$ dynes per sq. cm., nearly.
1 inch of mercury at 0° C. $=3\cdot388\times10^4$ dynes per sq. cm., nearly.

TABLE OF CONSTANTS.

The velocity acquired in falling for one second in vacuo, in any part of Great Britain, is about 32·2 feet per second, or 9·81 metres per second.

The pressure of one atmosphere, or 760 millimetres (29·922 inches) of mercury, is 1·033 kilogramme per sq. centimetre, or 14·73 lbs. per square inch.

The weight of a litre of dry air, at this pressure (at Paris) and 0° C., is 1·293 gramme.

The weight of a cubic centimetre of water is about 1 gramme.

The weight of a cubic foot of water is about 62·4 lbs.

The equivalent of a unit of heat, in gravitation units of energy, is—

772 for the foot and Fahrenheit degree.
1390 for the foot and Centigrade degree.
424 for the metre and Centigrade degree.
42400 for the centimetre and Centigrade degree.

In absolute units of energy, the equivalent is—

41·6 millions for the centimetre and Centigrade degree;

or 1 gramme-degree is equivalent to 41·6 million ergs.

ACOUSTICS.

CHAPTER LXII.

PRODUCTION AND PROPAGATION OF SOUND.

866. Sound is a Vibration.—Sound, as directly known to us by the sense of hearing, is an impression of a peculiar character, very broadly distinguished from the impressions received through the rest of our senses, and admitting of great variety in its modifications. The attempt to explain the physiological actions which constitute hearing forms no part of our present design. The business of physics is rather to treat of those external actions which constitute sound, considered as an objective existence external to the ear of the percipient.

Fig. 592.—Vibration of Straight Spring.

It can be shown, by a variety of experiments, that sound is the result of vibratory movement. Suppose, for example, we fix one end C of a straight spring C D (Fig. 592) in a vice A, then draw the other end D aside into the position D′, and let it go. In virtue of its elasticity (§ 126), the spring will return to its original position; but the kinetic energy which it acquires in returning is sufficient to carry it to a nearly equal distance on the other side; and it thus swings alternately from one side to the other through distances very gradually diminishing, until at last it comes to rest. Such movement is called vibratory. The motion from D′ to D″, or from D″ to D′, is called a *single vibration.* The two together constitute a

double or *complete vibration;* and the time of executing a complete vibration is the *period* of vibration. The *amplitude* of vibration for any point in the spring is the distance of its middle position from one of its extreme positions. These terms have been already employed (§ 107) in connection with the movements of pendulums to which indeed the movements of vibrating springs bear an extremely close resemblance. The property of isochronism, which approximately characterizes the vibrations of the pendulum, also belongs to the spring, the approximation being usually so close that the period may practically be regarded as altogether independent of the amplitude.

Fig. 593.—Vibration of Bell.

When the spring is long, the extent of its movements may generally be perceived by the eye. In consequence of the persistence of impressions, we see the spring in all its positions at once; and the edges of the space moved over are more conspicuous than the central parts, because the motion of the spring is slowest at its extreme positions.

As the spring is lowered in the vice, so as to shorten the vibrating portion of it, its movements become more rapid, and at the same time

more limited, until, when it is very short, the eye is unable to detect any sign of motion. But where sight fails us, hearing comes to our aid. As the vibrating part is shortened more and more, it emits a musical note, which continually rises in pitch; and this effect continues after the movements have become much too small to be visible.

It thus appears that a vibratory movement, if sufficiently rapid, may produce a sound. The following experiments afford additional illustration of this principle, and are samples of the evidence from which it is inferred that vibratory movement is essential to the production of sound.

Vibration of a Bell.—A point is fixed on a stand, in such a position as to be nearly in contact with a glass bell (Fig. 593). If a rosined fiddle-bow is then drawn over the edge of the bell, until a

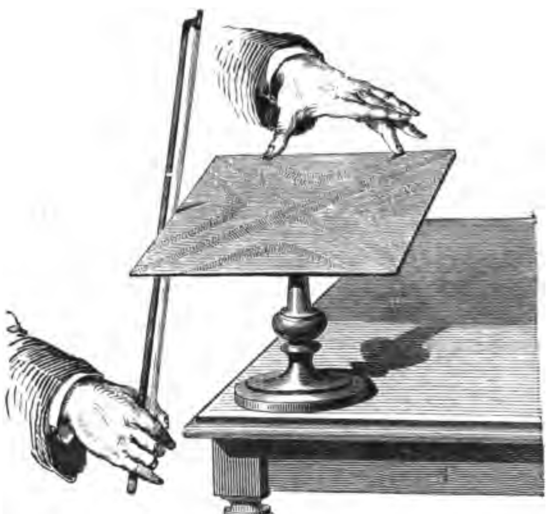


Fig. 594.—Vibration of Plate.

musical note is emitted, a series of taps are heard, due to the striking of the bell against the point. A pith-ball, hung by a thread, is driven out by the bell, and kept in oscillation as long as the sound continues. By lightly touching the bell, we may feel that it is vibrating; and if we press strongly, the vibration and the sound will both be stopped.

Vibration of a Plate.—Sand is strewn over the surface of a horizontal plate (Fig. 594), which is then made to vibrate by drawing a

bow over its edge. As soon as the plate begins to sound, the sand dances, leaves certain parts bare, and collects in definite lines, which are called *nodal lines*. These are, in fact, the lines which separate portions of the plate whose movements are in opposite directions. Their position changes whenever the plate changes its note.

Fig. 595. Vibration of String.

The vibratory condition of the plate is also manifested by another phenomenon, opposite—so to speak—to that just described. If very fine powder, such as lycopodium, be mixed with the sand, it will not move with the sand to the nodal lines, but will form little heaps in the centre of the vibrating segments; and these heaps will be in a state of violent agitation, with more or less of gyratory movement, as long as the plate is vibrating. This phenomenon, after long baffling explanation, was shown by Faraday to be due to indraughts of air, and ascending currents, brought about by the movements of the plate. In a moderately good vacuum, the lycopodium goes with the sand to the nodal lines.

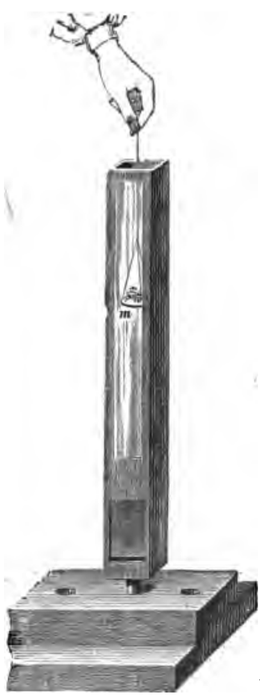


Fig. 596.—Vibration of Air.

Vibration of a String.—When a note is produced from a musical string or wire, its vibrations are often of sufficient amplitude to be detected by the eye. The string thus assumes the appearance of an elongated spindle (Fig. 595).

Vibration of the Air.—The sonorous body may sometimes be air, as in the case of organ-pipes, which we shall describe in a later chapter. It is easy to show by experiment that when a pipe *speaks*, the air within it is vibrating. Let one side of the tube be of glass, and let a small membrane *m*, stretched over a frame, be strewed with sand, and lowered into the pipe. The sand will be thrown into violent agitation, and the rattling of the grains, as they fall back on the membrane, is loud enough to be distinctly heard.

Singing Flames.—An experiment on the production of musical sound by flame, has long been known under the name of the *chemical harmonica.* An apperatus for the production of hydrogen gas (Fig. 597) is furnished with a tube, which tapers off nearly to a point at its upper end, where the gas issues and is lighted. When a tube, open at both ends, is held so as to surround the flame, a musical tone is heard, which varies with the dimensions of the tube, and often attains considerable power. The sound is due to the vibration of the air and products of combustion within the tube; and on observing the reflection of the flame in a mirror rotating about a vertical axis, it will be seen that the flame is alternately rising and falling, its successive images, as drawn out into a horizontal series by the rotation of the mirror, resembling a number of equidistant tongues of flame, with depressions between them. The experiment may also be performed with ordinary coal-gas.

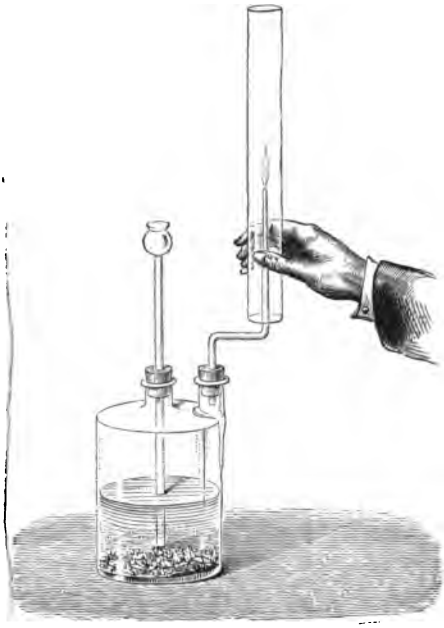


Fig. 597.—Chemical Harmonica.

Trevelyan Experiment.—A fire-shovel (Fig. 598) is heated, and balanced upon the edges of two sheets of lead fixed in a vice; it is then seen to execute a series of small oscillations—each end being alternately raised and depressed—and a sound is at the same time emitted. The oscillations are so small as to be scarcely perceptible in themselves; but they can be rendered very obvious by attaching to the shovel a small silvered mirror, on which a beam of light is directed. The reflected light can be made to form an image upon a screen, and this image is seen to be in a state of oscillation as long as the sound is heard.

The movements observed in this experiment are due to the sudden expansion of the cold lead. When the hot iron comes in contact with

a protuberance is instantly formed by dilatation, and the iron is thrown up. It then comes in contact with another portion of the

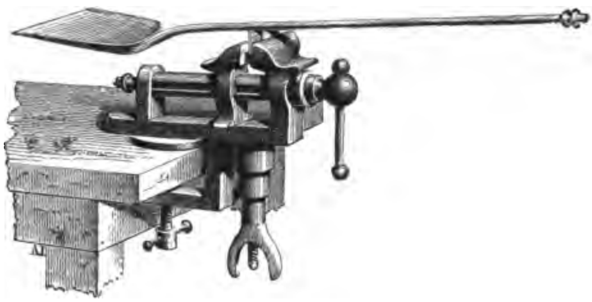


Fig. 598.—Trevelyan Experiment.

lead, where the same phenomenon is repeated while the first point cools. By alternate contacts and repulsions at the two points, the shovel is kept in a continual state of oscillation, and the regular succession of taps produces the sound.

The experiment is more usually performed with a special instrument invented by Trevelyan, and called a *rocker*, which, after being heated and laid upon a block of lead, rocks rapidly from side to side, and yields a loud note.

867. Distinctive Character of Musical Sound.—It is not easy to draw a sharp line of demarcation between musical sound and mere noise. The name of noise is usually given to any sound which seems unsuited to the requirements of music.

This unfitness may arise from one or the other of two causes. Either,

1. The sound may be unpleasant from containing discordant elements which jar with one another, as when several consecutive keys on a piano are put down together. Or,

2. It may consist of a confused succession of sounds, the changes being so rapid that the ear is unable to identify any particular note. This kind of noise may be illustrated by sliding the finger along a violin-string, while the bow is applied.

All sounds may be resolved into combinations of elementary musical tones occurring simultaneously and in succession. Hence the study of musical sounds must necessarily form the basis of acoustics.

Every sound which is recognized as musical is characterized by what may be called smoothness, evenness, or regularity; and the physical cause of this regularity is to be found in the accurate

periodicity of the vibratory movements which produce the sound. By *periodicity* we mean the recurrence of precisely similar states at equal intervals of time, so that the movements exactly repeat themselves; and the time which elapses between two successive recurrences of the same state is called the *period* of the movements.

Practically, musical and unmusical sounds often shade insensibly into one another. The tones of every musical instrument are accompanied by more or less of unmusical noise. The sounds of bells and drums have a sort of intermediate character; and the confused assemblage of sounds which is heard in the streets of a city blends at a distance into an agreeable hum.

868. Vehicle of Sound.—The origin of sound is always to be found in the vibratory movements of a sonorous body; but these vibratory movements cannot bring about the sensation of hearing unless there be a medium to transmit them to the auditory apparatus. This medium may be either solid, liquid, or gaseous, but it is necessary that it be elastic. A body vibrating in an absolute vacuum, or in a medium utterly destitute of elasticity, would fail to excite our sensations of hearing. This assertion is justified by the following experiments:—

1. Under the receiver of an air-pump is placed a clock-work arrangement for producing a number of strokes on a bell.

It is placed on a thick cushion of felt, or other inelastic material, and the air in the receiver is exhausted as completely as possible. If the clock-work is then started by means of the handle g, the hammer will be seen to strike the bell, but the sound will be scarcely audible. If hydrogen be introduced into the vacuum, and the receiver be again exhausted, the sound will be much more completely extinguished, being heard with difficulty even when the ear is placed in contact with the receiver. Hence it may fairly be concluded that if the receiver could be perfectly exhausted, and a perfectly inelastic support could be found for the bell, no sound at all would be emitted.

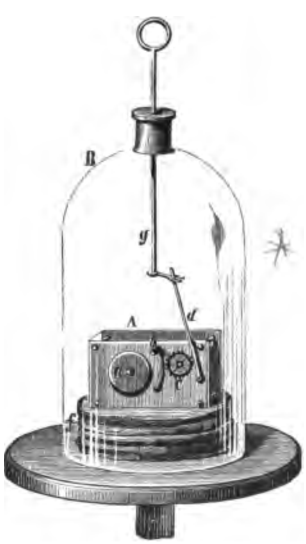


Fig. 599.—Sound in Exhausted Receiver.

2. The experiment may be varied by using a glass globe, furnished with a stop-cock, and having a little bell suspended within it by a thread. If the globe is exhausted of air, the sound of the bell will be scarcely audible. The globe may be filled with any kind of gas, or with vapour either saturated or non-saturated, and it will thus be found that all these bodies transmit sound.

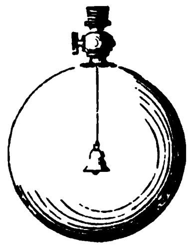


Fig. 600.
Globe with Stop-cock.

Sound is also transmitted through liquids, as may easily be proved by direct experiment. Experiment, however, is scarcely necessary for the establishment of the fact, seeing that fishes are provided with auditory apparatus, and have often an acute sense of hearing.

As to solids, some well-known facts prove that they transmit sound very perfectly. For example, light taps with the head of a pin on one end of a wooden beam, are distinctly heard by a person with his ear applied to the other end, though they cannot be heard at the same distance through the air. This property is sometimes employed as a test of the soundness of a beam, for the experiment will not succeed if the intervening wood is rotten, rotten wood being very inelastic.

The *stethoscope* is an example of the transmission of sound through solids. It is a cylinder of wood, with an enlargement at each end, and a perforation in its axis. One end is pressed against the chest of the patient, while the observer applies his ear to the other. He is thus enabled to hear the sounds produced by various internal actions, such as the beating of the heart and the passage of the air through the tubes of the lungs. Even simple *auscultation,* in which the ear is applied directly to the surface of the body, implies the transmission of sound through the walls of the chest.

By applying the ear to the ground, remote sounds can often be much more distinctly heard; and it is stated that savages can in this way obtain much information respecting approaching bodies of enemies.

We are entitled then to assert that *sound, as it affects our organs of hearing, is an effect which is propagated, from a vibrating body, through an elastic and ponderable medium.*

869. Mode of Propagation of Sound.—We will now endeavour to explain the action by which sound is propagated.

Let there be a plate *a* vibrating opposite the end of a long tube, and let us consider what happens during the passage of the plate

from its most backward position a'', to its most advanced position a'. This movement of the plate may be divided in imagination into a number of successive parts, each of which is communicated to the layer of air close in front of it, which is thus compressed, and, in its

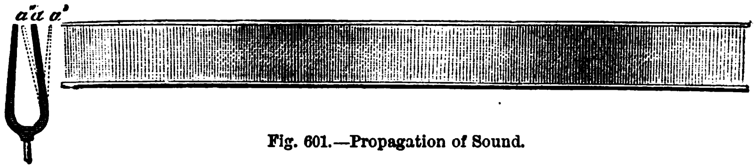


Fig. 601.—Propagation of Sound.

endeavour to recover from this compression, reacts upon the next layer, which is thus in its turn compressed. The compression is thus passed on from layer to layer through the whole tube, much in the same way as, when a number of ivory balls are laid in a row, if the first receives an impulse which drives it against the second, each ball will strike against its successor and be brought to rest.

The compression is thus passed on from layer to layer through the tube, and is succeeded by a rarefaction corresponding to the backward movement of the plate from a' to a''. As the plate goes on vibrating, these compressions and rarefactions continue to be propagated through the tube in alternate succession. The greatest compression in the layer immediately in front of the plate, occurs when the plate is at its middle position in its forward movement, and the greatest rarefaction occurs when it is in the same position in its backward movement. These are also the instants at which the plate is moving most rapidly.[1] When the plate is in its most advanced position, the layer of air next to it, A (Fig. 602) will be in its natural state, and another layer at A_1, half a wave-length further on, will also be in its natural state, the pulse having travelled from A to A_1, while the plate was moving from a'' to a'. At intervening points between A and A_1, the layers will have various amounts of compression corresponding to the different positions of the plate in its forward movement. The greatest compression is at C, a quarter of a wave-length in advance of A, having travelled over this distance while the plate was advancing from a to a'. The compressions at D and D_1 repre-

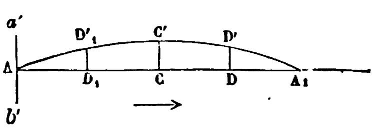


Fig. 602.—Graphical Representation.

[1] See § 870, also Note A at the end of this chapter.

sent those which existed immediately in front of the plate when it had advanced respectively one-fourth and three-fourths of the distance from a'' to a', and the curve $A\,C'\,A_1$ is the graphical representation both of condensation and velocity for all points in the air between A and A_1.

If the plate ceased vibrating, the condition of things now existing in the portion of air $A\,A_1$ would be transferred to successive portions of air in the tube, and the curve $A\,C'\,A_1$ would, as it were, slide onward through the tube with the velocity of sound, which is about 1100 feet per second. But the plate, instead of remaining permanently at a', executes a backward movement, and produces rarefactions and retrograde velocities, which are propagated onwards in the same manner as the condensations and forward velocities. A complete wave of the undulation is accordingly represented by the curve $A\,E'\,A_1\,C'\,A_2$ (Fig. 603), the portions of the curve below the line of

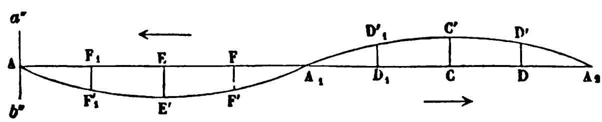


Fig. 603.—Graphical Representation of Complete Wave.

abscissas being intended to represent rarefactions and retrograde velocities. If we suppose the vibrating plate to be rigidly connected with a piston which works air-tight in the tube, the velocities of the particles of air in the different points of a wave-length will be identical with the velocities of the piston at the different parts of its motion.

The wave-length $A\,A_2$ is the distance that the pulse has travelled while the vibrating plate was moving from its most backward to its most advanced position, and back again. During this time, which is called the *period* of the vibrations, each particle of air goes through its complete cycle of changes, both as regards motion and density. The period of vibration of any particle is thus identical with that of the vibrating plate, and is the same as the time occupied by the waves in travelling a wave-length. Thus, if the plate be one leg of a common A tuning-fork, making 435 complete vibrations per second, the period will be $\frac{1}{435}$th of a second, and the undulation will travel in this time a distance of $\frac{1100}{435}$ feet, or 2 feet 6 inches, which is therefore the wave-length in air for this note. If the plate continues to vibrate in a uniform manner, there will be a continual series of equal

and similar waves running along the tube with the velocity of sound. Such a succession of waves constitutes an undulation. Each wave consists of a condensed portion, and a rarefied portion, which are distinguished from each other in Fig. 601 by different tints, the dark shading being intended to represent condensation.

870. Nature of Undulations.—The possibility of condensations and rarefactions being propagated continually in one direction, while each particle of air simply moves backwards and forwards about its original position, is illustrated by Fig. 604, which represents, in an

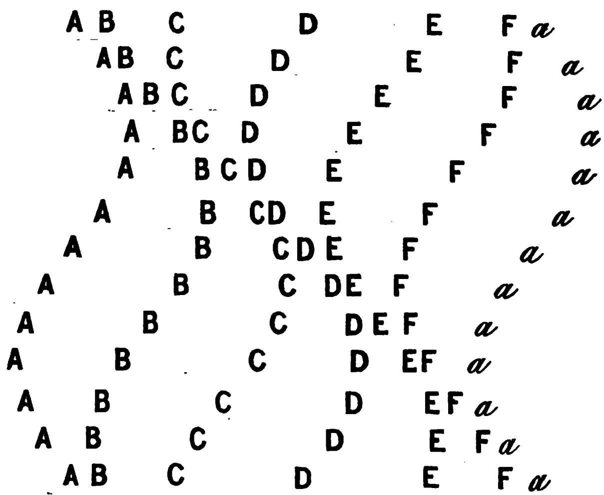


Fig. 604.—Longitudinal Vibration.

exaggerated form, the successive phases of an undulation propagated through 7 particles A B C D E F *a* originally equidistant, the distance from the first to the last being one wave-length of the undulation. The diagram is composed of thirteen horizontal rows, the first and last being precisely alike. The successive rows represent the positions of the particles at successive times, the interval of time from each row to the next being $\frac{1}{12}$th of the period of the undulation.

In the first row A and *a* are centres of condensation, and D is a centre of rarefaction. In the third row B is a centre of condensation, and E a centre of rarefaction. In the fifth row the condensation and rarefaction have advanced by one more letter, and so on through the whole series, the initial state of things being

reproduced when each of these centres has advanced through a wave-length, so that the thirteenth row is merely a repetition of the first.

The velocities of the particles can be estimated by the comparison of successive rows. It is thus seen that the greatest forward velocity is at the centres of condensation, and the greatest backward velocity at the centres of rarefaction. Each particle has its greatest velocities, and greatest condensation and rarefaction, in passing through its mean position, and comes for an instant to rest in its positions of greatest displacement, which are also positions of mean density.

The distance between A and a remains invariable, being always a wave-length, and these two particles always agree in phase. Any other two particles represented in the diagram are always in different phases, and the phases of A and D, or B and E, or C and F, are always opposite; for example, when A is moving forwards with the maximum velocity, D is moving backwards with the same velocity.

The vibrations of the particles, in an undulation of this kind, are called *longitudinal;* and it is by such vibrations that sound is propagated through air. Fig. 605 illustrates the manner in which an undulation may be propagated by means of *transverse* vibrations, that is to say, by vibrations executed in a direction perpendicular to that in which the undulation advances. Thirteen particles A B C D E F G H I J K L a are represented in the positions which they occupy at successive times, whose interval is one-sixth of a period. At the instant first considered, D and J are the particles which are furthest displaced. At the end of the first interval, the wave has advanced two letters, so that F and L are now the furthest displaced. At the end of the next interval, the wave has advanced two letters further, and so on, the state of things at the end of the six intervals, or of one complete period, being the same as at the beginning, so that the seventh line is merely a repetition of the first. Some examples of this kind of wave-motion will be mentioned in later chapters.

871. Propagation in an Open Space.—When a sonorous disturbance occurs in the midst of an open body of air, the undulations to which it gives rise run out in all directions from the source. If the disturbance is symmetrical about a centre, the waves will be spherical; but this case is exceptional. A disturbance usually produces condensation on one side, at the same instant that it produces rarefaction on another. This is the case, for example, with a vibrating

plate, since, when it is moving towards one side, it is moving away from the other. These inequalities which exist in the neighbourhood of the sonorous body, have, however, a tendency to become less marked, and ultimately to disappear, as the distance is increased. Fig. 606 represents a diametral section of a series of spherical waves. Their mode of propagation has some analogy to that of the circular

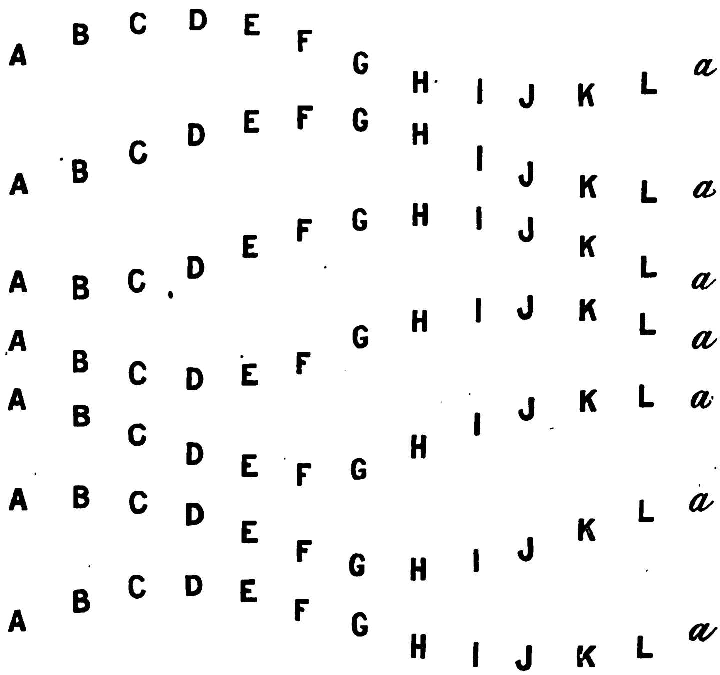


Fig. 605.—Transverse Vibration.

waves produced on water by dropping a stone into it; but the particles which form the waves of water rise and fall, whereas those which form sonorous waves merely advance and retreat, their lines of motion being always coincident with the directions along which the sound travels. In both cases it is important to remark that *the undulation does not involve a movement of transference.* Thus, when the surface of a liquid is traversed by waves, bodies floating on it rise and fall, but are not carried onward. This property is characteristic of undulations generally. *An undulation may be defined as a system of movements in which the several particles move to and fro, or round and round, about definite points, in such a*

manner as to produce the continued onward transmission of a condition, or series of conditions.

There is one important difference between the propagation of sound in a uniform tube and in an open space. In the former case, the layers of air corresponding to successive wave-lengths are of equal

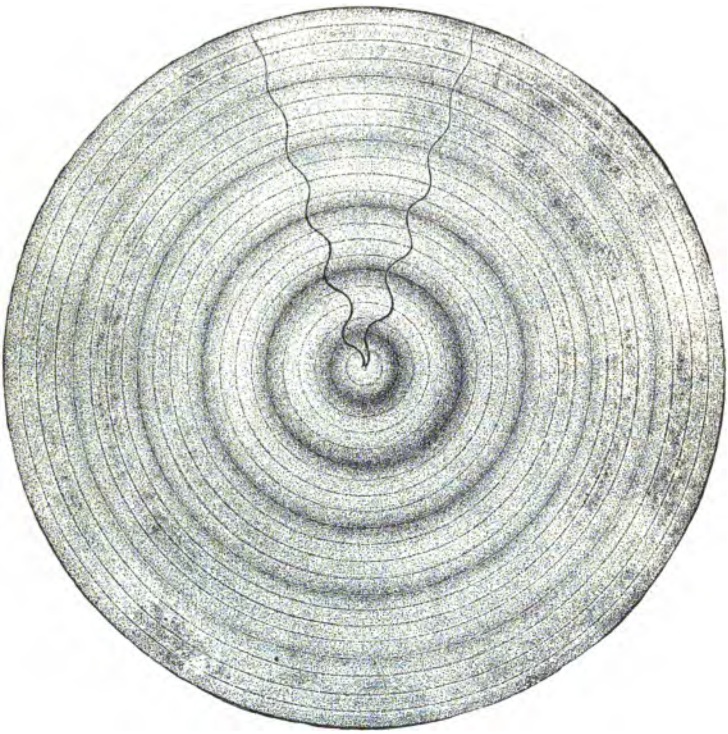


Fig. 606.—Propagation in Open Space.

mass, and their movements are precisely alike, except in so far as they are interfered with by friction. Hence sound is transmitted through tubes to great distances with but little loss of intensity, especially if the tubes are large.[1]

The same principle is illustrated by the ease with which a scratch

[1] Regnault, in his experiments on the velocity of sound, found that in a conduit ·108 of a metre in diameter, the report of a pistol charged with a gramme of powder ceased to be heard at the distance of 1150 metres. In a conduit of ·3^{m}, the distance was 3810^{m}. In the great conduit of the St. Michel sewer, of 1^{m}·10, the sound was made by successive reflections to traverse a distance of 10,000 metres without becoming inaudible.—*D.*

on a log of wood is heard at the far end, the substance of the log acting like the body of air within a tube.

In an open space, each successive layer has to impart its own condition to a larger layer; hence there is a continual diminution of amplitude in the vibrations as the distance from the source increases. This involves a continual decrease of loudness. An undulation involves the onward transference of energy; and the amount of energy which traverses, in unit time, any closed surface described about the source, must be equal to the energy which the source emits in unit time. Hence, by the reasoning which we employed in the case of radiant heat (§ 465), it follows that the intensity of sonorous energy diminishes according to the law of inverse squares.

The energy of a particle executing simple vibrations in obedience to forces of elasticity, varies as the square of the amplitude of its excursions; for, if the amplitude be doubled, the distance worked through, and the mean working force, are both doubled, and thus the work which the elastic forces do during the movement from either extreme position to the centre is quadrupled. This work is equal to the energy of the particle in any part of its course. At the extreme positions it is all in the shape of potential energy; in the middle position it is all in the shape of kinetic energy; and at intermediate points it is partly in one of these forms, and partly in the other.

It can be shown that exactly half the energy of a complete wave is kinetic, the other half being potential.

872. Dissipation of Sonorous Energy.—The reasoning by which we have endeavoured to establish the law of inverse squares, assumes that onward propagation involves no loss of sonorous energy. This assumption is not rigorously true, inasmuch as vibration implies friction, and friction implies the generation of heat, at the expense of the energy which produces the vibrations. Sonorous energy must therefore diminish with distance somewhat more rapidly than according to the law of inverse squares. All sound, in becoming extinct, becomes converted into heat.

This conversion is greatly promoted by defect of homogeneity in the medium of propagation. In a fog, or a snow-storm, the liquid or solid particles present in the air produce innumerable reflections, in each of which a little sonorous energy is converted into heat.

873. Velocity of Sound in Air.—The propagation of sound through an elastic medium is not instantaneous, but occupies a very sensible time in traversing a moderate distance. For example, the flash of a gun at the distance of a few hundred yards is seen some time before the report is heard. The interval between the two impressions may be regarded as representing the time required for the propagation of the sound across the intervening distance, for the time occupied by the propagation of light across so small a distance is inappreciable.

It is by experiments of this kind that the velocity of sound in air has been most accurately determined. Among the best determinations may be mentioned that of Lacaille, and other members of a commission appointed by the French Academy in 1738; that of Arago, Bouvard, and other members of the Bureau de Longitudes in 1822; and that of Moll, Vanbeek, and Kuytenbrouwer in Holland, in the same year. All these determinations were obtained by firing cannon at two stations, several miles distant from each other, and noting, at each station, the interval between seeing the flash and hearing the sound of the guns fired at the other. If guns were fired only at one station, the determination would be vitiated by the effect of wind blowing either with or against the sound. The error from this cause is nearly eliminated by firing the guns alternately at the two stations, and still more completely by firing them simultaneously This last plan was adopted by the Dutch observers, the distance of the two stations in their case being about nine miles. Regnault has quite recently repeated the investigation, taking advantage of the important aid afforded by modern electrical methods for registering the times of observed phenomena. All the most careful determinations agree very closely among themselves, and show that the velocity of sound through air at 0° C. is about 332 metres, or 1090 feet per second.[1] The velocity increases with the temperature, being proportional to *the square root of the absolute temperature by air thermometer* (§ 325). If t denote the ordinary Centigrade tempera-

[1] A recent determination by Mr. Stone at the Cape of Good Hope is worthy of note as being based on the comparison of observations made through the sense of hearing alone. It had previously been suggested that the two senses of sight and hearing, which are concerned in observing the flash and report of a cannon, might not be equally prompt in receiving impressions (Airy on *Sound*, p. 131). Mr. Stone accordingly placed two observers —one near a cannon, and the other at about three miles distance; each of whom on hearing the report, gave a signal through an electric telegraph. The result obtained was in precise agreement with that stated in the text.

ture, and a the coefficient of expansion ·00366, the velocity of sound through air at any temperature is given by the formula

$$332\sqrt{1+at} \text{ in metres per second, or}$$
$$1090\sqrt{1+at} \text{ in feet per second.}$$

The actual velocity of sound from place to place on the earth's surface is found by compounding this velocity with the velocity of the wind.

There is some reason, both from theory and experiment, for believing that very loud sounds travel rather faster than sounds of moderate intensity.

874. Theoretical Computation of Velocity.—By applying the principles of dynamics to the propagation of undulations,[1] it is computed that the velocity of sound through air must be given by the formula

$$v=\sqrt{\frac{E}{D}}. \qquad (1)$$

D denoting the density of the air, and E its coefficient of elasticity, as measured by the quotient of pressure applied by compression produced.

Let P denote the pressure of the air in units of force per unit of area; then, if the temperature be kept constant during compression, a small additional pressure p will, by Boyle's law, produce a compression equal to $\frac{p}{P}$, and the value of E, being the quotient of p by this quantity, will be simply P.

On the other hand, if no heat is allowed either to enter or escape, the temperature of the air will be raised by compression, and additional resistance will thus be encountered. In this case, as shown in § 500, the coefficient of elasticity will be Pk, k denoting the ratio of the two specific heats, which for air and simple gases is about 1·41.

It thus appears that the velocity of sound in air cannot be less than $\sqrt{\frac{P}{D}}$ nor greater than $\sqrt{1{\cdot}41\,\frac{P}{D}}$. Its actual velocity, as determined by observation, is identical, or practically identical, with the latter of these limiting values. Hence we must infer that the compressions and extensions which the particles of air undergo in transmitting sound are of too brief duration to allow of any sensible transference of heat from particle to particle.

This conclusion is confirmed by another argument due to Professor

[1] See note B at the end of this chapter.

Stokes. If the inequalities of temperature due to compression and expansion were to any sensible degree smoothed down by conduction and radiation, this smoothing down would diminish the amount of energy available for wave-propagation, and would lead to a falling off in intensity incomparably more rapid than that due to the law of inverse squares.

875. Numerical Calculation.—The following is the actual process of calculation for perfectly dry air at 0° C., the centimetre, gramme, and second being taken as the units of length, mass, and time.

The density of dry air at 0°, under the pressure of 1033 grammes per square centimetre, at Paris, is ·001293 of a gramme per cubic centimetre. But the gravitating force of a gramme at Paris is 981 dynes (§ 91). The density ·001293 therefore corresponds to a pressure of 1033×981 dynes per sq. cm.; and the expression for the velocity in centimetres per second is

$$v = \sqrt{1 \cdot 41 \frac{P}{D}} = \sqrt{1 \cdot 41 \frac{1033 \times 981}{\cdot 001293}} = 33240 \text{ nearly};$$

that is, 332·4 metres per second, or 1093 feet per second.

876. Effects of Pressure, Temperature, and Moisture.—The velocity of sound is independent of the height of the barometer, since changes of this element (at constant temperature) affect P and D in the same direction, and to the same extent.

For a given density, if P_0 denote the pressure at 0°, and α the coefficient of expansion of air, the pressure at $t°$ Centigrade is $P_0(1+\alpha t)$, the value of α being about $\frac{1}{273}$.

Hence, if the velocity at 0° be 1090 feet per second, the velocity at $t°$ will be $1090\sqrt{1+\frac{t}{273}}$. At the temperature 50° F. or 10° C., which is approximately the mean annual temperature of this country, the value of this expression is about 1110, and at 86° F. or 30° C. it is about 1148. The increase of velocity is thus about a foot per second for each degree Fahrenheit.

The humidity of air has some influence on the velocity of sound, inasmuch as aqueous vapour is lighter than air; but the effect is comparatively trifling, at least in temperate climates. At the temperature 50° F., air saturated with moisture is less dense than dry air by about 1 part in 220, and the consequent increase of velocity cannot be greater than about 1 part in 440, which will be between 2 and 3 feet per second. The increase should, in fact, be somewhat

less than this, inasmuch as the value of k (the ratio of the two specific heats) appears to be only 1·31 for aqueous vapour.[1]

877. Newton's Theory, and Laplace's Modification.—The earliest theoretical investigation of the velocity of sound was that given by Newton in the *Principia* (book 2, section 8). It proceeds on the tacit assumption that no changes of temperature are produced by the compressions and extensions which enter into the constitution of a sonorous undulation; and the result obtained by Newton is equivalent to the formula

$$v=\sqrt{\frac{P}{D}};$$

or since (§ 210) $\frac{P}{D}=gH$, where H denotes the *height of a homogeneous atmosphere,* and the velocity acquired in falling through any height s is $\sqrt{2gs}$, the velocity of sound in air is, according to Newton, the same as *the velocity which would be acquired by falling in vacuo through half the height of a homogeneous atmosphere.* This, in fact, is the form in which Newton states his result.[2]

Newton himself was quite aware that the value thus computed theoretically was too small, and he throws out a conjecture as to the cause of the discrepancy; but the true cause was first pointed out by Laplace, as depending upon increase of temperature produced by compression, and decrease of temperature produced by expansion.

878. Velocity in Gases generally.—The same principles which apply to air apply to gases generally; and since for all simple gases the ratio of the two specific heats is 1·41, the velocity of sound in any simple gas is $\sqrt{1{\cdot}41\frac{P}{D}}$, D denoting its absolute density at the pressure P. Comparing two gases at the same pressure, we see that the velocities of sound in them will be inversely as the square roots of their absolute densities; and this will be true whether the temperatures of the two gases are the same or different.

879. Velocity of Sound in Liquids.—The velocity of sound in water was measured by Colladon, in 1826, at the Lake of Geneva. Two boats were moored at a distance of 13,500 metres (between 8 and 9 miles). One of them carried a bell, weighing about 140 lbs., immersed in the lake. Its hammer was moved by an external lever, so arranged as to ignite a small quantity of gunpowder at the instant

[1] Rankine on the *Steam Engine,* p. 320.

[2] Newton's investigation relates only to *simple* waves; but if these have all the same velocity (as Newton shows), this must also be the velocity of the complex wave which they compose. Hence the restriction is only apparent.

of striking the bell. An observer in the other boat was enabled to hear the sound by applying his ear to the extremity of a trumpet-shaped tube (Fig. 607), having its lower end covered with a membrane and facing towards the direction from which the sound proceeded. By noting the interval between seeing the flash and hearing the sound, the velocity with which the sound travelled through the water was determined. The velocity thus computed was 1435 metres per second, and the temperature of the water was 8°·1 C.

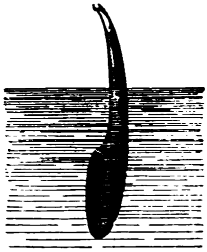


Fig. 607.

Formula (1) of § 874 holds for liquids as well as for gases.

The resistance of water to compression is about $2 \cdot 1 \times 10^{10}$ dynes per sq. cm., and the correcting factor for the heat of compression, as calculated by § 505, is 1·0012, which may be taken as unity. The density is also unity. Hence we have

$$v = \sqrt{\frac{E}{D}} = \sqrt{(2 \cdot 1 \times 10^{10})} = 144914 \text{ cm. per sec.},$$

that is, about 1449 metres per second; which agrees sufficiently well with the experimental determination.

Wertheim has measured the velocity of sound in some liquids by an indirect method, which will be explained in a later chapter. He finds it to be 1160 metres per second in ether and alcohol, and 1900 in a solution of chloride of calcium.

880. Velocity of Sound in Solids.—The velocity of sound in cast-iron was determined by Biot and Martin by means of a connected series of water-pipes, forming a conduit of a total length of 951 metres. One end of the conduit was struck with a hammer, and an observer at the other end heard two sounds, the first transmitted by the metal, and the second by the air, the interval between them being 2·5 seconds. Now the time required for travelling this distance through air, at the temperature of the experiment (11° C.), is 2·8 seconds. The time of transmission through the metal was therefore ·3 of a second, which is at the rate of 3170 metres per second. It is, however, to be remarked, that the transmitting body was not a continuous mass of iron, but a series of 376 pipes, connected together by collars of lead and tarred cloth, which must have considerably delayed the transmission of the sound. But in spite of this, the velocity is about nine times as great as in air.

Wertheim, by the indirect methods above alluded to, measured the velocity of sound in a number of solids, with the following results, the velocity in air being taken as the unit of velocity:—

Lead,	3·974 to 4·120	Steel,	14·361 to 15·108
Tin,	7·338 to 7·480	Iron,	15·108
Gold,	5·603 to 6·424	Brass,	10·224
Silver,	7·903 to 8·057	Glass,	14·956 to 16·759
Zinc,	9·863 to 11·009	Flint Glass,	11·890 to 12·220
Copper,	11·167	Oak,	9·902 to 12·02
Platinum,	7·823 to 8·467	Fir,	12·49 to 17·26

881. Theoretical Computation.—The formula $\sqrt{\frac{E}{D}}$ serves for solids as well as for liquids and gases; but as solids can be subjected to many different kinds of strain, whereas liquids and gases can be subjected to only one, we may have different values of E, and different velocities of transmission of pulses for the same solid. This is true even in the case of a solid whose properties are alike in all directions (called an *isotropic* solid); but the great majority of solids are very far from fulfilling this condition, and transmit sound more rapidly in some directions than in others.

When the sound is propagated by alternate compressions and extensions running along a substance which is not prevented from extending and contracting laterally, the elasticity E becomes identical[1] with Young's modulus (§ 128). On the other hand, if uniform spherical waves of alternate compression and extension spread outwards, symmetrically, from a point in the centre of an infinite solid, lateral extension and contraction will be prevented by the symmetry of the action. The effective elasticity is, in this case, greater than Young's modulus, and the velocity of sound will be increased accordingly.

By the table on p. 79 the value of Young's modulus for copper is 120×10^{10}, and by the table on p. xii. the density of copper is about 8·8. Hence, for the velocity of sound through a copper rod, in centimetres per second, we have

$$v = \sqrt{\frac{E}{D}} = \sqrt{\frac{120 \times 10^{10}}{8 \cdot 8}} = 369300 \text{ nearly,}$$

or 3693 metres per second.

This is about 11·1 times the velocity in air.

[1] Subject to a very small correction for heat of compression, which can be calculated by the formula of § 505. In the case of iron, the correcting factor is about 1·0023.

882. Reflection of Sound.—When sonorous waves meet a fixed obstacle they are reflected, and the two sets of waves—one direct, and the other reflected—are propagated just as if they came from two separate sources. If the reflecting surface is plane, waves di-

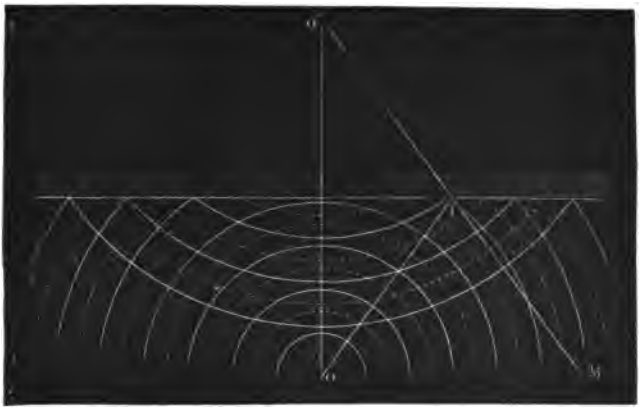


Fig. 608.—Reflection of Sound.

verging from any centre O (Fig. 608) in front of it are reflected so as to diverge from a centre O′ symmetrically situated behind it, and an ear at any point M in front hears the reflected sound as if it came from O′.

The direction from which a sound appears to the hearer to proceed is determined by the direction along which the sonorous pulses are propagated, and is always normal to the waves. A normal to a set of sound-waves may therefore conveniently be called a *ray* of sound.

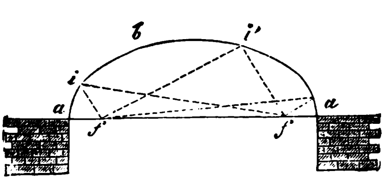


Fig. 609.—Reflection from Elliptic Roof.

O I is a direct ray, and I M the corresponding reflected ray; and it is obvious, from the symmetrical position of the points O O′, that these two rays are equally inclined to the surface, or *the angles of incidence and reflection are equal.*

883. Illustrations of Reflection of Sound.—The reflection of sonorous waves explains some well-known phenomena. If *aba* (Fig. 609) be an elliptic dome or arch, a sound emitted from either of the foci *ff* will be reflected from the elliptic surface in such a direction as to pass through the other focus. A sound emitted from either focus

may thus be distinctly heard at the other, even when quite inaudible at nearer points. This is a consequence of the property, that lines drawn to any point on an ellipse from the two foci are equally inclined to the curve.

The experiment of the conjugate mirrors (§ 468) is also applicable to sound. Let a watch be hung in the focus of one of them (Fig. 610),

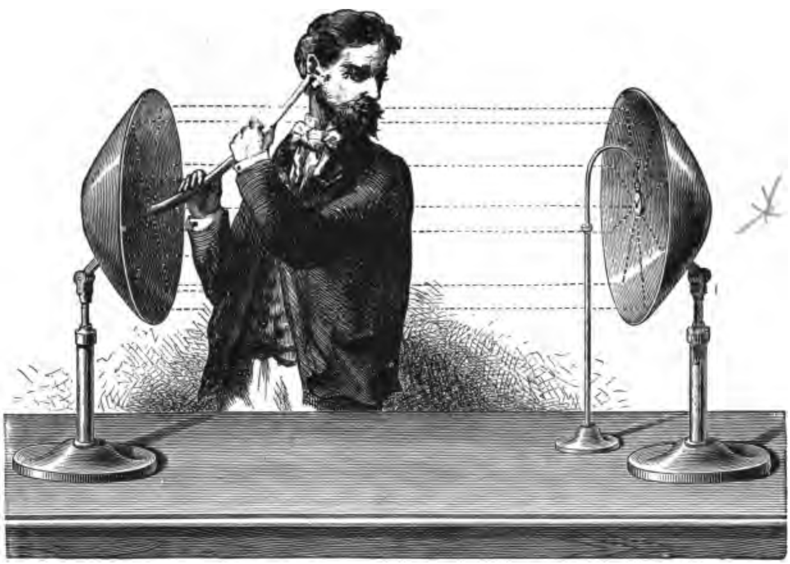


Fig. 610.—Reflection of Sound from Conjugate Mirrors.

and let a person hold his ear at the focus of the other; or still better, to avoid intercepting the sound before it falls on the second mirror, let him employ an ear-trumpet, holding its further end at the focus. He will distinctly hear the ticking, even when the mirrors are many yards apart.[1]

884. Echo.—Echo is the most familiar instance of the reflection of sound. In order to hear the echo of one's own voice, there must be a distant body capable of reflecting sound directly back, and the number of syllables that an echo will repeat is proportional to the

[1] Sondhaus has shown that sound, like light, is capable of being *refracted*. A spherical balloon of collodion, filled with carbonic acid gas, acts as a sound-lens. If a watch be hung at some distance from it on one side, an ear held at the conjugate focus on the other side will hear the ticking. See also a later section on "Curved Rays of Sound" in the chapter on the "Wave Theory of Light."

distance of this obstacle. The sounds reflected to the speaker have travelled first over the distance O A (Fig. 611) from him to the reflecting body, and then back from A to O. Supposing five syllables to be pronounced in a second, and taking the velocity of sound as 1100 feet per second, a distance of 550 feet from the speaker to the reflecting body would enable the speaker to complete the fifth syllable before the return of the first; this is at the rate of 110 feet

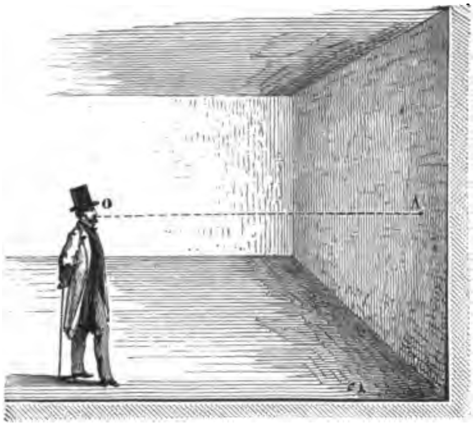


Fig. 611.—Echo.

per syllable. At distances less than about 100 feet there is not time for the distinct reflection of a single syllable; but the reflected sound mingles with the voice of the speaker. This is particularly observable under vaulted roofs.

Multiple echoes are not uncommon. They are due, in some cases, to independent reflections from obstacles at different distances; in others, to reflections of reflections. A position exactly midway between two parallel walls, at a sufficient distance apart, is favourable for the observance of this latter phenomenon. One of the most frequently cited instances of multiple echoes is that of the old palace of Simonetta, near Milan, which forms three sides of a quadrangle. According to Kircher, it repeats forty times.

885. Speaking and Hearing Trumpets.—The complete explanation of the action of these instruments presents considerable difficulty. The speaking-trumpet (Fig. 612) consists of a long tube (sometimes 6 feet long), slightly tapering towards the speaker, furnished at this end with a hollow mouth-piece, which nearly fits the lips, and at

the other with a funnel-shaped enlargement, called the *bell*, opening out to a width of about a foot. It is much used at sea, and is found very effectual in making the voice heard at a distance. The explanation usually given of its action is, that the slightly conical form of the long tube produces a series of reflections in directions more and more nearly parallel to the axis; but this explanation fails to account for the utility of the *bell*, which experience has shown to be considerable. It appears from a theoretical investigation by Lord Rayleigh that the speaking-trumpet causes a greater total quantity of sonorous energy to be produced from the same expenditure of breath.[1]

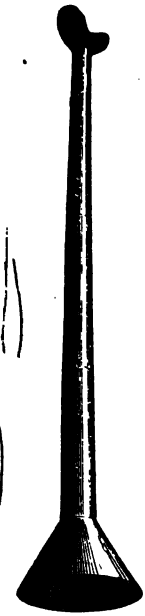


Fig. 612. Speaking-trumpet.

Ear-trumpets have various forms, as represented in Fig. 613; having little in common, except that the external opening or *bell* is much larger than the end which is introduced into the ear. Membranes of gold-beaters' skin are sometimes stretched across their interior, in the positions indicated by the dotted lines in Nos. 4 and 5. No. 6 consists simply of a bell with such a membrane stretched across its outer end, while its inner end communicates with the ear by an indian-rubber tube with an ivory end-piece. These light membranes are peculiarly susceptible of impression from aerial vibrations. In Regnault's experiments above cited, it was found that membranes were affected at distances greater than those at which sound was heard.

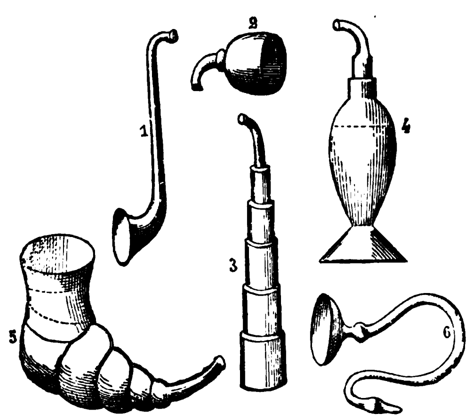


Fig. 613.—Ear-trumpets.

886. Interference of Sonorous Undulations.—When two systems of waves are traversing the same matter, the actual motion of each particle of the matter is the resultant of the motions due to each system separately. When these component motions are in the same direction the resultant is their

[1] *Theory of Sound*, vol. ii. p. 102.

sum; when they are in opposite directions it is their difference; and if they are equal, as well as opposite, it is zero. Very remarkable phenomena are thus produced when the two undulations have the same, or nearly the same wave-length; and the action which occurs in this case is called *interference.*

When two sonorous undulations of exactly equal wave-length and amplitude are traversing the same matter in the same direction, their phases must either be the same, or must everywhere differ by the same amount. If they are the same, the amplitude of vibration for each particle will be double of that due to either undulation separately. If they are opposite—in other words, if one undulation be half a wave-length in advance of the other—the motions which they would separately produce in any particle are equal and opposite, and the particle will accordingly remain at rest. Two sounds will thus, by their conjoint action, produce silence.

In order that the extinction of sound may be complete, the rarefied portions of each set of waves must be the *exact* counterparts of the condensed portions of the other set, a condition which can only be approximately attained in practice.

The following experiment, due to M. Desains, affords a very direct illustration of the principle of interference. The bottom of a wooden box is pierced with an opening, in which a powerful whistle fits. The top of the box has two larger openings symmetrically placed with respect to the lower one. The inside of the box is lined with felt, to prevent the vibrations from being communicated to the box, and to weaken internal reflection. When the whistle is sounded, if a membrane, with sand strewn on it, is held in various positions in the vertical plane which bisects, at right angles, the line joining the two openings, the sand will be agitated, and will arrange itself in nodal lines. But if it is carried out of this plane, positions will be found, at equal distances on both sides of it, at which the agitation is scarcely perceptible. If, when the membrane is in one of these positions, we close one of the two openings, the sand is again agitated, clearly showing that the previous absence of agitation was due to the interference of the undulations proceeding from the two orifices.

In this experiment the proof is presented to the eye. In the following experiment, which is due to M. Lissajous, it is presented to the ear. A circular plate, supported like the plate in Fig. 594, is made to vibrate in sectors separated by radial nodes. The number of sectors will always be even, and adjacent sectors will vibrate

in opposite directions. Let a disk of card-board of the same size be divided into the same number of sectors, and let alternate sectors be cut away, leaving only enough near the centre to hold the remaining sectors together. If the card be now held just over the vibrating disk, in such a manner that the sectors of the one are exactly over those of the other, a great increase of loudness will be observed, consequent on the suppression of the sound from alternate sectors; but if the card-board disk be turned through the width of half a sector, the effect no longer occurs. If the card is made to rotate rapidly in a continuous manner, the alterations of loudness will form a series of beats.

It is for a similar reason that, when a large bell is vibrating, a person in its centre hears the sound as only moderately loud, while within a short distance of some portions of the edge the loudness is intolerable.

887. Interference of Direct and Reflected Waves.[1] Nodes and Antinodes.—Interference may also occur between undulations travelling in opposite directions; for example, between a direct and a reflected system. When waves proceeding along a tube meet a rigid obstacle, forming a cross section of the tube, they are reflected directly back again, the motion of any particle close to the obstacle being compounded of that due to the direct wave, and an equal and opposite motion due to the reflected wave. The reflected waves are in fact the images (with reference to the obstacle regarded as a plane mirror) of the waves which would exist in the prolongation of the tube if the obstacle were withdrawn. At the distance of half a wave-length from the obstacle the motions due to the direct and reflected waves will accordingly be equal and opposite, so that the particles situated at this distance will be permanently at rest; and the same is true at the distance of any number of half wave-lengths from the obstacle. The air in the tube will thus be divided into a number of vibrating segments separated by nodal planes or cross sections of no vibration arranged at distances of half a wave-length apart. One of these nodes is at the obstacle itself. At the centres of the vibrating segments—that is to say, at the distance of a quarter wave-length *plus* any number of half wave-lengths from the obstacle or from any node —the velocities due to the direct and reflected waves will be equal and in the same direction, and the amplitude of vibration will accordingly be double of that due to the direct wave alone. These

[1] See note C, page 895.

are the sections of greatest disturbance as regards change of place. We shall call them *antinodes*. On the other hand, it is to be remembered that motion *with* the direct wave is motion *against* the reflected waves, and *vice versa*, so that (§ 869) at points where the velocities due to both have the same absolute direction they correspond to condensation in the case of one of these undulations, and to rarefaction in the case of the other. Accordingly, these sections of maximum movement are the places of no change of density; and on the other hand, the nodes are the places where the changes of density are greatest. If the reflected undulation is feebler than the direct one, as will be the case, for example, if the obstacle is only imperfectly rigid, the destruction of motion at the nodes and of change of density at the antinodes will not be complete; the former will merely be places of minimum motion, and the latter of minimum change of density.

Direct experiments in verification of these principles, a wall being the reflecting body, were conducted by Savart, and also by Seebeck, the latter of whom employed a testing apparatus called the acoustic pendulum. It consists essentially of a small membrane stretched in a frame, from the top of which hangs a very light pendulum, with its bob resting against the centre of the membrane. In the middle portions of the vibrating segments the membrane, moving with the air on its two faces, throws back the pendulum, while it remains nearly free from vibration at the nodes.

Regnault made extensive use of the acoustic pendulum in his experiments on the velocity of sound. The pendulum, when thrown back by the membrane, completed an electric circuit, and thus effected a record of the instant when the sound arrived.

888. Beats Produced by Interference.—When two notes which are not quite in unison are sounded together, a peculiar palpitating effect is produced;—we hear a series of bursts of sound, with intervals of comparative silence between them. The bursts of sound are called *beats*, and the notes are said to *beat* together. If we have the power of tuning one of the notes, we shall find that as they are brought more nearly into unison, the beats become slower, and that, as the departure from unison is increased, the beats become more rapid, till they degenerate first into a rattle, and then into a discord. The effect is most striking with deep notes.

These beats are completely explained by the principle of interference. As the wave-lengths of the two notes are slightly different,

while the velocity of propagation is the same, the two systems of waves will, in some portions of their course, agree in phase, and thus strengthen each other; while in other parts they will be opposite in phase, and will thus destroy each other. Let one of the notes, for example, have 100 vibrations per second, and the other 101. Then, if we start from an instant when the maxima of condensation from the two sources reach the ear together, the next such conjunction will occur exactly a second later. During the interval the maxima of one system have been gradually falling behind those of the other, till, at the end of the second, the loss has amounted to one wave-length. At the middle of the second it will have amounted to half a wave-length, and the two sounds will destroy each other. We shall thus have one beat and one extinction in each second, as a consequence of the fact that the higher note has made one vibration more than the lower. In general, the frequency of beats is the difference of the frequencies of vibration of the beating notes.

NOTE A. § 869.

That the particles which are moving forward are in a state of compression, may be shown in the following way:—Consider an imaginary cross section travelling forward through the tube with the same velocity as the undulation. Call this velocity v, and the velocity of any particle of air u. Also let the density of any particle be denoted by ρ. Then u and ρ remain constant for the imaginary moving section, and the mass of air which it traverses in its motion per unit time is $(v-u)\,\rho$. As there is no permanent transfer of air in either direction through the tube, the mass thus traversed must be the same as if the air were at rest at its natural density. Hence the value of $(v-u)\,\rho$ is the same for all cross sections; whence it follows, that where u is greatest ρ must be greatest, and where u is negative ρ is less than the natural density.

If ρ_0 denote the natural density, we have $(v-u)\,\rho=v\,\rho_0$, whence $\frac{u}{v}=\frac{\rho-\rho_0}{\rho}$; that is to say, *the ratio of the velocity of a particle to the velocity of the undulation is equal to the condensation existing at the particle.* If u is negative—that is to say, if the velocity be retrograde —its ratio to v is a measure of the rarefaction.

From this principle we may easily derive a formula for the velocity of sound, bearing in mind that u is always very small in comparison with v.

For, consider a thin lamina of air whose thickness is δx, and let δu, $\delta \rho$, and δp be the excesses of the velocity, density, and pressure on the second side of the lamina above those on the first at the same moment. The above equation, $(v-u)\,\rho=v\,\rho_0$, gives $(v-u)\,\delta\rho-\rho\,\delta u=0$, whence $\frac{\delta u}{\delta \rho}=\frac{v-u}{\rho}$, or, since u may be neglected in comparison with v,

$$\frac{\delta u}{\delta \rho}=\frac{v}{\rho}.$$

The time which the moving section occupies in traversing the lamina is $\frac{\delta x}{v}$, and in this time the velocity of the lamina changes by the amount $-\delta u$, since the velocity on the

second side of the lamina is $u+\delta u$ at the beginning and u at the end of the time. The force producing this change of velocity (if the section of the tube be unity) is $-\delta p$, or $-1{\cdot}41\,\frac{p}{\rho}\,\delta\rho$, and must be equal to the quotient of change of momentum by time, that is to $-\rho\,\delta x\,.\,\delta u \div \frac{\delta x}{v}$, or to $-\rho v\,\delta u$. Hence $\frac{\delta u}{\delta\rho}=1{\cdot}41\,\frac{p}{\rho^2 v}$. Equating this to the other expression for $\frac{\delta u}{\delta\rho}$, we have

$$\frac{v}{\rho}=1{\cdot}41\,\frac{p}{\rho^2 v} \qquad , \qquad v^2=1{\cdot}41\,\frac{p}{\rho}.$$

This investigation is due to Professor Rankine, *Phil. Trans.* 1869.

Note B. § 874.

The following is the usual investigation of the velocity of transmission of sound through a uniform tube filled with air, friction being neglected: Let x denote the original distance of a particle of air from the section of the tube at which the sound originates, and $x+y$ its distance at time t, so that y is the displacement of the particle from the position of equilibrium. Then a particle which was originally at distance $x+\delta x$ will at time t be at the distance $x+\delta x+y+\delta y$; and the thickness of the intervening lamina, which was originally δx, is now $\delta x+\delta y$. Its compression is therefore $-\frac{\delta y}{\delta x}$ or ultimately $-\frac{dy}{dx}$, and if P denote the original pressure, the increase of pressure is $-1{\cdot}41\,\mathrm{P}\,\frac{dy}{dx}$. The excess of pressure behind a lamina δx above the pressure in front is $\frac{d}{dx}\left(1{\cdot}41\,\mathrm{P}\,\frac{dy}{dx}\right)\delta x$, or $1{\cdot}41\,\mathrm{P}\,\frac{d^2y}{dx^2}\,\delta x$; and if D denote the original density of the air, the acceleration of the lamina will be the quotient of this expression by $\mathrm{D}\,.\,\delta x$. But this acceleration is $\frac{d^2y}{dt^2}$. Hence we have the equation

$$\frac{d^2y}{dt^2}=1{\cdot}41\,\frac{\mathrm{P}}{\mathrm{D}}\,\frac{d^2y}{dx^2};$$

the integral of which is

$$y=\mathrm{F}\,(x-vt)+f\,(x+vt);$$

where v denotes $\sqrt{1{\cdot}41\,\frac{\mathrm{P}}{\mathrm{D}}}$, and F, f denote any functions whatever.

The term $\mathrm{F}\,(x-vt)$ represents a wave, of the form $y=\mathrm{F}\,(x)$, travelling forwards with velocity v; for it has the same value for $t_1+\delta t$ and $x_1+v\,.\,\delta t$ as for t_1 and x_1. The term $f\,(x+vt)$ represents a wave, of the form $y=f\,(x)$, travelling backwards with the same velocity.

In order to adapt this investigation, as well as that given in Note A, to the propagation of longitudinal vibrations through any elastic material, whether solid, liquid, or gaseous, we have merely to introduce E in the place of 1·41 P, E denoting the coefficient of elasticity of the substance, as defined by the condition that a compression $\frac{dy}{dx}$ is produced by a force (per unit area) of $\mathrm{E}\,\frac{dy}{dx}$.

Note C. § 887.

The following is the regular mathematical investigation of the interference of direct and reflected waves of the simplest type, in a uniform tube.

Using x, y, and t in the same sense as in Note B, and measuring x from the reflecting surface to meet the incident waves, we have, for the incident waves,

$$y_1 = a \sin \frac{x+vt}{\lambda} 2\pi, \qquad (1)$$

a denoting the amplitude, and λ the wave-length. For the reflected waves, we have

$$y_2 = a \sin \frac{x-vt}{\lambda} 2\pi, \qquad (2)$$

since this equation represents waves equal and opposite to the former, and satisfies the condition that at the reflecting surface (where x is zero) the total disturbance $y_1 + y_2$ is zero. Putting y for $y_1 + y_2$, we have, by adding the above equations and employing a well-known formula of trigonometry,

$$y = 2a \sin \frac{x}{\lambda} 2\pi \,.\, \cos \frac{vt}{\lambda} 2\pi. \qquad (3)$$

The extension (or compression if negative) is $\frac{dy}{dx}$, and we have

$$\frac{dy}{dx} = \frac{2\pi}{\lambda} 2a \cos \frac{x}{\lambda} 2\pi \,.\, \cos \frac{vt}{\lambda} 2\pi. \qquad (4)$$

The factor $\sin \frac{x}{\lambda} 2\pi$ vanishes at the points for which x is either zero or a multiple of $\frac{1}{2}\lambda$, and attains its greatest values (in arithmetical sense) at those for which x is $\frac{1}{4}\lambda$, or $\frac{1}{4}\lambda$ *plus* a multiple of $\frac{1}{2}\lambda$. On the other hand, the factor $\cos \frac{x}{\lambda} 2\pi$ vanishes at the latter points, and attains its greatest values at the former. The points for which $\sin \frac{x}{\lambda} 2\pi$ vanishes are the nodes, since at these points y is constantly zero; and the points for which $\cos \frac{x}{\lambda} 2\pi$ vanishes are the antinodes, since at these the extension or compression is constantly zero.

The motion represented by equation (3) is the simplest type of ***stationary undulation.***

Note D. § 888.

The following is the mathematical investigation of beats for two systems of waves of equal amplitude but slightly different wave-length and period, travelling with the same velocity.

Denote $x - vt$ by θ, $\frac{2\pi}{\lambda_1}$ by m_1, and $\frac{2\pi}{\lambda_2}$ by m_2, λ_1 λ_2 being the wave-lengths of the two systems, and let their common amplitude be a. Then the resultant of the two sets is represented by

$$\begin{aligned} y &= a \sin m_1\theta + a \sin m_2\theta \\ &= 2a \sin \tfrac{1}{2}(m_1 + m_2)\theta \,.\, \cos \tfrac{1}{2}(m_1 - m_2)\theta. \end{aligned}$$

By hypothesis $m_1 - m_2$ is very small compared with $m_1 + m_2$; hence the factor $\cos \frac{1}{2}(m_1 - m_2)\theta$ remains nearly constant for an increment of θ which causes $\frac{1}{2}(m_1 + m_2)\theta$ to increase by 2π. The expression therefore represents a series of waves having a wave-length intermediate between λ_1 and λ_2 (since $\frac{1}{2}(m_1 + m_2)$ is intermediate between m_1 and m_2), and having an amplitude $2a \cos \frac{1}{2}(m_1 - m_2)\theta$ which gradually varies between the limits zero and $2a$.

CHAPTER LXIII.

NUMERICAL EVALUATION OF SOUND.

889. Qualities of Musical Sound.—Musical tones differ one from another in respect of three qualities;—loudness, pitch, and character.

Loudness.—The loudness of a sound considered subjectively is the intensity of the sensation with which it affects the organs of hearing. Regarded objectively, it depends, in the case of sounds of the same pitch and character, upon the energy of the aerial vibrations in the neighbourhood of the ear, and is proportional to the square of the amplitude.

Our auditory apparatus is, however, so constructed as to be more susceptible of impression by sounds of high than of low pitch. A bass note must have much greater energy of vibration than a treble note, in order to strike the ear as equally loud. The intensity of sonorous vibration at a point in the air is therefore not an absolute measure of the intensity of the sensation which will be received by an ear placed at the point.

The word loud is also frequently applied to a source of sound, as when we say a loud voice, the reference being to the loudness as heard at a given distance from the source. The diminution of loudness with increase of distance according to the law of inverse squares is essentially connected with the proportionality of loudness to square of amplitude.

Pitch.—Pitch is the quality in respect of which an acute sound differs from a grave one; for example, a treble note from a bass note. All persons are capable of appreciating differences of pitch to some extent, and the power of forming accurate judgments of pitch constitutes what is called a *musical ear*.

Physically, pitch depends solely on *frequency of vibration*, that is to say, on the number of vibrations executed per unit time. In

ordinary circumstances this frequency is the same for the source of sound, the medium of transmission, and the drum of the ear of the person hearing; and in general the transmission of vibrations from one body or medium to another produces no change in their frequency. The *second* is universally employed as the unit of time in treating of sonorous vibrations; so that *frequency* means *number of vibrations per second.* Increase of frequency corresponds to elevation of pitch.

Period and *frequency* are reciprocals. For example, if the period of each vibration is $\frac{1}{100}$ of a second, the number of vibrations per second is 100. Period therefore is an absolute measure of pitch, and the longer the period the lower is the pitch.

The wave-length of a note in any medium is the distance which sound travels in that medium during the period corresponding to the note. Hence wave-length may be taken as a measure of pitch, provided the medium be given; but, in passing from one medium to another, wave-length varies directly as the velocity of sound. The wave-length of a given note in air depends upon the temperature of the air, and is shortened in transmission from the heated air of a concert-room to the colder air outside, while the pitch undergoes no change.

If we compare a series of notes rising one above another by what musicians regard as equal differences of pitch, their frequencies will not be equidifferent, but will form an increasing geometrical progression, and their periods (and wave-lengths in a given medium) will form a decreasing geometrical progression.

Character.—Musical sounds may, however, be alike as regards pitch and loudness, and may yet be easily distinguishable. We speak of the *quality* of a singer's voice, and the *tone* of a musical instrument; and we characterize the one or the other as rich, sweet, or mellow; on the one hand: or as poor, harsh, nasal, &c., on the other. These epithets are descriptive of what musicians call *timbre*—a French word literally signifying *stamp.* German writers on acoustics denote the same quality by a term signifying *sound-tint.* It might equally well be called *sound-flavour.* We adopt *character* as the best English designation.

Physically considered, as wave-length and wave-amplitude fall under the two previous heads, *character* must depend upon the only remaining point in which aerial waves can differ—namely their *form,* meaning by this term the law according to which the velo-

cities and densities change from point to point of a wave. This subject will be more fully treated in Chapter lxv. Every musical sound is more or less mingled with non-musical noises, such as puffing, scraping, twanging, hissing, rattling, &c. These are not comprehended under *timbre* or *character* in the usage of the best writers on acoustics. The gradations of loudness which characterize the commencement, progress, and cessation of a note, and upon which musical effect often greatly depends, are likewise excluded from this designation. In distinguishing the sounds of different musical instruments, we are often guided as much by these gradations and extraneous accompaniments as by the character of the musical tones themselves.

890. Musical Intervals.—When two notes are heard, either simultaneously or in succession, the ear experiences an impression of a special kind, involving a perception of the relation existing between them as regards difference of pitch. This impression is often recognized as identical where absolute pitch is very different, and we express this identity of impression by saying that the *musical interval* is the same.

Each musical interval, thus recognized by the ear as constituting a particular relation between two notes, is found to correspond to a particular *ratio* between their frequencies of vibration. Thus the *octave*, which of all intervals is that which is most easily recognized by the ear, is the relation between two notes whose *frequencies* are as 1 to 2, the upper note making twice as many vibrations as the lower in any given time.

It is the musician's business so to combine sounds as to awaken emotions of the peculiar kind which are associated with works of art. In attaining this end he employs various resources, but musical intervals occupy the foremost place. It is upon the judicious employment of these that successful composition mainly depends.

891. Gamut.—The *gamut* or *diatonic scale* is a series of eight notes having certain definite relations to one another as regards frequency of vibration. The first and last of the eight are at an interval of an octave from each other, and are called by the same name; and by taking in like manner the octaves of the other notes of the series, we obtain a repetition of the gamut both upwards and downwards, which may be continued over as many octaves as we please.

The notes of the gamut are usually called by the names

Do	Re	Mi	Fa	Sol	La	Si	Do_2

and their vibration-frequencies are proportional to the numbers

1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2

or, clearing fractions, to

24	27	30	32	36	40	45	48

The intervals from Do to each of the others in order are called a *second,* a *major third,* a *fourth,* a *fifth,* a *sixth,* a *seventh,* and an *octave* respectively. The interval from La to Do_2 is called a *minor third,* and is evidently represented by the ratio $\frac{6}{5}$.

The interval from Do to Re, from Fa to Sol, or from La to Si, is represented by the ratio $\frac{9}{8}$, and is called a *major tone.* The interval from Re to Mi, or from Sol to La, is represented by the ratio $\frac{10}{9}$, and is called a *minor tone.* The interval from Mi to Fa, or from Si to Do_2, is represented by the ratio $\frac{16}{15}$, and is called a *limma.* As the square of $\frac{16}{15}$ is a little greater than $\frac{9}{8}$, a limma is rather more than half a major tone.

The intervals between the successive notes of the gamut are accordingly represented by the following ratios[1]:—

Do	Re	Mi	Fa	Sol	La	Si	Do_2
$\frac{9}{8}$	$\frac{10}{9}$	$\frac{16}{15}$	$\frac{9}{8}$	$\frac{10}{9}$	$\frac{9}{8}$	$\frac{16}{15}$	

Do (with all its octaves) is called the *key-note* of the piece of music, and may have any pitch whatever. In order to obtain perfect harmony, the above ratios should be accurately maintained whatever the key-note may be.

892. Tempered Gamut.—A great variety of keys are employed in music, and it is a practical impossibility, at all events in the case of instruments like the piano and organ, which have only a definite set of notes, to maintain these ratios strictly for the whole range of possible key-notes. Compromise of some kind becomes necessary, and different systems of compromise are called different *temperaments* or different *modes of temperament.* The temperament which is most in favour in the present day is the simplest possible, and is called *equal temperament,* because it favours no key above another, but makes the tempered gamut exactly the same for all. It ignores the

[1] The logarithmic differences, which are accurately proportional to the intervals, are approximately as under, omitting superfluous zeros.

Do	Re	Mi	Fa	Sol	La	Si	Do
51	46	28	51	46	51	28	

difference between major and minor tones, and makes the limma exactly half of either. The interval from Do to Do_2 is thus divided into 5 tones and 2 semitones, a tone being $\frac{1}{6}$ of an octave, and a semitone $\frac{1}{12}$ of an octave. The ratio of frequencies corresponding to a tone will therefore be the sixth root of 2, and for a semitone it will be the 12th root of 2.

The difference between the natural and the tempered gamut for the key of C is shown by the following table, which gives the number of complete vibrations per second for each note of the middle octave of an ordinary piano:—

	Tempered Gamut.	Natural Gamut.		Tempered Gamut.	Natural Gamut.
C . .	258·7	258·7	G . .	387·6	388·0
D . .	290·3	291·0	A . .	435·0	431·1
E . .	325·9	323·4	B . .	488·2	485·0
F . .	345·3	344·9	C . .	517·3	517·3

The absolute pitch here adopted is that of the Paris Conservatoire, and is fixed by the rule that A (the middle A of a piano, or the A string of a violin) is to have 435 complete vibrations per second in the tempered gamut. This is rather lower than the concert-pitch which has prevailed in this country in recent years, but is probably not so low as that which prevailed in the time of Handel. It will be noted that the number of vibrations corresponding to C is approximately equal to a power of 2 (256 or 512). Any power of 2 accordingly expresses (to the same degree of approximation) the number of vibrations corresponding to one of the octaves of C.

The Stuttgard congress (1834) recommended 528 vibrations per second for C, and the C tuning-forks sold under the sanction of the Society of Arts are guaranteed to have this pitch. By multiplying the numbers 24, 27 . . . 48, in § 891, by 11, we shall obtain the frequencies of vibration for the natural gamut in C corresponding to this standard. What is generally called *concert-pitch* gives C about 538. The C of the Italian Opera is 546. Handel's C is said to have been $499\frac{1}{2}$.

893. Limits of Pitch employed in Music.—The deepest note regularly employed in music is the C of 32 vibrations per second which is emitted by the longest pipe (the 16-foot pipe) of most organs. Its wave-length in air at a temperature at which the velocity of sound is 1120 feet per second, is $\frac{1120}{32}=35$ feet. The highest note employed seldom exceeds A, the third octave of the A above defined. Its number of vibrations per second is $435\times2^3=3480$, and

its wave-length in air is about 4 inches. Above this limit it is difficult to appreciate pitch, but notes of at least ten times this number of vibrations are audible.

The average compass of the human voice is about two octaves. The deep F of a bass-singer has 87, and the upper G of the treble 775 vibrations per second. Voices which exceed either of these limits are regarded as deep or high.

894. Minor Scale and Pythagorean Scale.—The difference between a major and minor tone is expressed by the ratio $\frac{81}{80}$, and is called a *comma*. The difference between a minor tone and a limma is expressed by the ratio $\frac{25}{24}$, and is the smallest value that can be assigned to the somewhat indefinite interval denoted by the name *semitone*, the greatest value being the limma itself ($\frac{16}{15}$). The signs ♯ and ♭ (sharp and flat) appended to a note indicate that it is to be raised or lowered by a semitone. The major scale or gamut, as above given, is modified in the following way to obtain the minor scale:—

Do		Re		Mi♭		Fa		Sol		La♭		Si♭		Do$_2$
	$\frac{9}{8}$		$\frac{16}{15}$		$\frac{10}{9}$		$\frac{9}{8}$		$\frac{16}{15}$		$\frac{9}{8}$		$\frac{10}{9}$	

the numbers in the second line being the ratios which represent the intervals between the successive notes.

It is worthy of note that Pythagoras, who was the first to attempt the numerical evaluation of musical intervals, laid down a scheme of values slightly different from that which is now generally adopted. According to him, the intervals between the successive notes of the major scale are as follows:—

Do		Re		Mi		Fa		Sol		La		Si		Do
	$\frac{9}{8}$		$\frac{9}{8}$		$\frac{256}{243}$		$\frac{9}{8}$		$\frac{9}{8}$		$\frac{9}{8}$		$\frac{256}{243}$	

This scheme agrees exactly with the common system as regards the values of the fourth, fifth, and octave, and makes the values of the major third, the sixth, and the seventh each greater by a comma, while the small interval from *mi* to *fa*, or from *si* to *do*, is diminished by a comma. In the ordinary system, the prime numbers which enter the ratios are 2, 3, and 5; in the Pythagorean system they are only 2 and 3; hence the interval between any two notes of the Pythagorean scale can be expressed as the sum or difference of a certain number of octaves and fifths. In tuning a violin by making the intervals between the strings true fifths, the Pythagorean scheme is virtually employed.

895. Methods of Counting Vibrations. Siren.—The instrument which is chiefly employed for counting the number of vibrations corresponding to a given note, is called the *siren*, and was devised by Cagniard de Latour. It is represented in Figs. 614, 615, the former being a front, and the latter a back view.

There is a small wind-chest, nearly cylindrical, having its top pierced with fifteen holes, disposed at equal distances round the circumference of a circle. Just over this, and nearly touching it, is a movable circular plate, pierced with the same number of holes

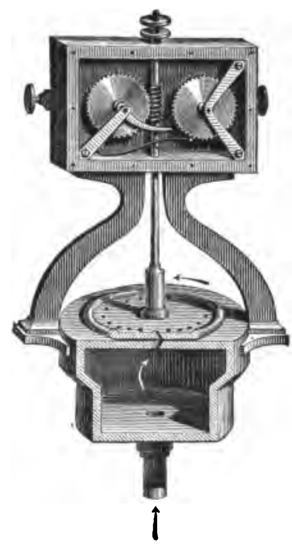


Fig. 614. Siren. Fig. 615.

similarly arranged, and so mounted that it can rotate very freely about its centre, carrying with it the vertical axis to which it is attached. This rotation is effected by the action of the wind, which enters the wind-chest from below, and escapes through the holes. The form of the holes is shown by the section in Fig. 615. They do not pass perpendicularly through the plates, but slope contrary ways, so that the air when forced through the holes in the lower plate impinges *upon one side* of the holes in the upper plate, and thus blows it round in a definite direction. The instrument is driven by means of the bellows shown in Fig. 625 (§ 910). As the rotation of one plate upon the other causes the holes to be alternately opened and closed, the wind escapes in successive puffs, whose frequency

depends upon the rate of rotation. Hence a note is emitted which rises in pitch as the rotation becomes more rapid.

The siren will sound under water, if water is forced through it instead of air; and it was from this circumstance that it derived its name.

In each revolution, the fifteen holes in the upper plate come opposite to those in the lower plate 15 times, and allow the compressed air in the wind-chest to escape; while in the intervening positions its escape is almost entirely prevented. Each revolution thus gives rise to 15 vibrations; and in order to know the number of vibrations corresponding to the note emitted, it is only necessary to have a means of counting the revolutions.

This is furnished by a counter, which is represented in Fig. 615. The revolving axis carries an endless screw, driving a wheel of 100 teeth, whose axis carries a hand traversing a dial marked with 100 divisions. Each revolution of the perforated plate causes this hand to advance one division. A second toothed-wheel is driven intermittently by the first, advancing suddenly one tooth whenever the hand belonging to the first wheel passes the zero of its scale. This second wheel also carries a hand traversing a second dial; and at each of the sudden movements just described this hand advances one division. Each division accordingly indicates 100 revolutions of the perforated plate, or 1500 vibrations. By pushing in one of the two buttons which are shown, one on each side of the box containing the toothed-wheels, we can instantaneously connect or disconnect the endless screw and the first toothed-wheel.

In order to determihe the number of vibrations corresponding to any given sound which we have the power of maintaining steadily, we fix the siren on the bellows, the screw and wheel being disconnected, and drive the siren until the note which it emits is judged to be in unison with the given note. We then, either by regulating the pressure of the wind, or by employing the finger to press with more or less friction against the revolving axis, contrive to keep the note of the siren constant for a measured interval of time, which we observe by a watch. At the commencement of the interval we suddenly connect the screw and toothed-wheel, and at its termination we suddenly disconnect them, having taken care to keep the siren in unison with the given sound during the interval. As the hands do not advance on the dials when the screw is out of connection with the wheels, the readings before and after the measured interval of

time can be taken at leisure. Each reading consists of four figures, indicating the number of revolutions from the zero position, units and tens being read off on the first dial, and hundreds and thousands on the second. The difference of the two readings is the number of revolutions made in the measured interval, and when multiplied by 15 gives the number of vibrations in the interval, whence the number of vibrations per second is computed by division.

896. Graphic Method.—In the hands of a skilful operator, with a good musical ear, the siren is capable of yielding very accurate determinations, especially if, by adding or subtracting the number of beats,

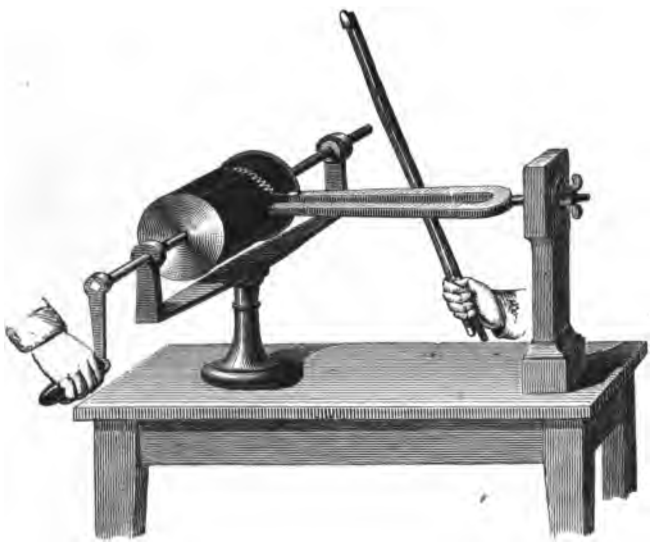


Fig. 616.—Vibroscope.

correction be made for any slight difference of pitch between the siren and the note under investigation.

The vibrations of a tuning-fork can be counted, without the aid of the siren, by a graphical method, which does not call for any exercise of musical judgment, but simply involves the performance of a mechanical operation.

The tuning-fork is fixed in a horizontal position, as shown in Fig. 616, and has a light style, which may be of brass wire, quill, or bristle, attached to one of its prongs, by wax or otherwise. To receive the trace, a piece of smoked paper is gummed round a cylinder, which can be turned by a handle, a screw cut on the axis

causing it at the same time to travel endwise. The cylinder is placed so that the style barely touches the blackened surface. The fork is then made to vibrate by bowing it, and the cylinder is turned. The result is a wavy line traced on the blackened surface, and the number of wave-forms (each including a pair of bends in opposite directions) is the number of vibrations. If the experiment lasts for a measured interval of time, we have only to count these wave-forms, and divide by the number of seconds, in order to obtain the number of vibrations per second for the note of the tuning-fork. By plunging the paper in ether, the trace will be fixed, so that the paper may be laid aside, and the vibrations counted at leisure. The apparatus is called the *vibroscope*, and was invented by Duhamel.

M. Léon Scott has invented an instrument called the *phonautograph*, which is adapted to the graphical representation of sounds in

Fig. 617.—Traces by Phonautograph.

general. The style, which is very light, is attached to a membrane stretched across the smaller end of what may be called a large ear-trumpet. The membrane is agitated by the aerial waves proceeding from any source of sound, and the style leaves a record of these agitations on a blackened cylinder, as in Duhamel's apparatus. Fig. 617 represents the traces thus obtained from the sound of a tuning-fork in three different modes of vibration.

897. Tonometer.—When we have determined the frequency of vibration for a particular tuning-fork, that of another fork, nearly in unison with it, can be deduced by making the two forks vibrate simultaneously, and counting the beats which they produce.

Scheibler's *tonometer*, which is constructed by Koenig of Paris, consists of a set of 65 tuning-forks, such that any two consecutive forks make 4 beats per second, and consequently differ in pitch by

4 vibrations per second. The lowest of the series makes 256 vibrations, and the highest 512, thus completing an octave. Any note within this range can have its vibration-frequency at once determined, with great accuracy, by making it sound simultaneously with the fork next above or below it, and counting beats.

With the aid of this instrument, a piano can be tuned with certainty to any desired system of temperament, by first tuning the notes which come within the compass of the tonometer, and then proceeding by octaves.

In the ordinary methods of tuning pianos and organs, temperament is to a great extent a matter of chance; and a tuner cannot attain the same temperament in two successive attempts.

898. Pitch modified by Relative Motion.—We have stated in § 889 that, in ordinary circumstances, the frequency of vibration in the source of sound, is the same as in the ear of the listener, and in the intervening medium. This identity, however, does not hold if the source of sound and the ear of the listener are approaching or receding from each other. Approach of either to the other produces increased frequency of the pulses on the ear, and consequent elevation of pitch in the sound as heard; while recession has an opposite effect. Let n be the number of vibrations performed in a second by the source of the sound, v the velocity of sound in the medium, and a the relative velocity of approach. Then the number of waves which reach the ear of the listener in a second, will be n *plus* the number of waves which cover a length a, that is (since n waves cover a length v), will be $n+\frac{a}{v}n$ or $\frac{v+a}{v}n$.

The following investigation is more rigorous. Let the source make n vibrations per second. Let the observer move towards the source with velocity a. Let the source move away from the observer with velocity a'. Let the medium move from the observer towards the source with velocity m, and let the velocity of sound in the medium be v.

Then the velocity of the observer relative to the medium is $a-m$ towards the source, and the velocity of the source relative to the medium is $a'-m$ away from the observer. The velocity of the sound relative to the source will be different in different directions, its greatest amount being $v+a'-m$ towards the observer, and its least being $v-a'+m$ away from the observer. The length of a wave will vary with direction, being $\frac{1}{n}$ of the velocity of the sound

relative to the source. The length of those waves which meet the observer will be $\frac{v+a'-m}{n}$, and the velocity of these waves relative to the observer will be $v+a-m$; hence the number of waves that meet him in a second will be $\frac{v+a-m}{v+a'-m}n$.

Careful observation of the sound of a railway whistle, as an express train dashes past a station, has confirmed the fact that the sound as heard by a person standing at the station is higher while the train is approaching than when it is receding. A speed of about 40 miles an hour will sharpen the note by a semitone in approaching, and flatten it by the same amount in receding, the natural pitch being heard at the instant of passing.[1]

[1] The best observations of this kind were those of Buys Ballot, in which trumpeters, with their instruments previously tuned to unison, were stationed, one on the locomotive, and others at three stations beside the line of railway. Each trumpeter was accompanied by musicians, charged with the duty of estimating the difference of pitch between the note of his trumpet and those of the others, as heard before and after passing.

CHAPTER LXIV.

MODES OF VIBRATION.

899. Longitudinal and Transverse Vibrations of Solids.—Sonorous vibrations are manifestations of elasticity. When the particles of a solid body are displaced from their natural positions relative to one another by the application of external force, they tend to return, in virtue of the elasticity of the body. When the external force is removed, they spring back to their natural position, pass it in virtue of the velocity acquired in the return, and execute isochronous vibrations about it until they gradually come to rest. The isochronism of the vibrations is proved by the constancy of pitch of the sound emitted; and from the isochronism we can infer, by the aid of mathematical reasoning, that the restoring force increases directly as the displacement of the parts of the body from their natural relative position (§ 111).

The same body is, in general, susceptible of many different modes of vibration, which may be excited by applying forces to it in different ways. The most important of these are comprehended under the two heads of *longitudinal* and *transverse* vibrations.

In the former the particles of the body move to and fro in the direction along which the pulses travel, which is always regarded as the longitudinal direction, and the deformations produced consist in alternate compressions and extensions. In the latter the particles move to and fro in directions transverse to that in which the pulses travel, and the deformation consists in bending. To produce longitudinal vibrations, we must apply force in the longitudinal direction. To produce transverse vibration, we must apply force transversely.

900. Transverse Vibrations of Strings.—To the transverse vibrations of strings, instrumental music is indebted for some of its most

precious resources. In the violin, violoncello, &c., the strings are set in vibration by drawing a bow across them. The part of the bow which acts on the strings consists of hairs tightly stretched and rubbed with rosin. The bow adheres to the string, and draws it aside till the reaction becomes too great for the adhesion to overcome. As the bow continues to be drawn on, slipping takes place, and the mere fact of slipping diminishes the adhesion. The string accordingly springs back suddenly through a finite distance. It is then again caught by the bow, and the same action is repeated. In the harp and guitar, the strings are plucked with the finger, and then left to vibrate freely. In the piano the wires are struck with little hammers faced with leather. The pitch of the sound emitted in these various cases depends only on the string itself, and is the same whichever mode of excitation be employed.

901. Laws of the Transverse Vibrations of Strings.—It can be shown by an investigation closely analogous to that which gives the velocity of sound in air, that the velocity with which transverse vibrations travel along a perfectly flexible string is given by the formula

$$v = \sqrt{\frac{t}{m}}; \qquad (1)$$

t denoting the tension of the string, and m the mass of unit length of it. If m be expressed in grammes per centimetre of length, t should be expressed in dynes (§ 87), and the value obtained for v will be in centimetres per second. The sudden disturbance of any point in the string, causes two pulses to start from this point, and run along the string in opposite directions. Each of these, on arriving at the end of the free portion of the string, is reflected from the solid support to which the string is attached, and at the same time undergoes reversal as to side. It runs back, thus reversed, to the other end of the free portion, and there again undergoes reflection and reversal. When it next arrives at the origin of the disturbance it has travelled over just twice the length of the string; and as this is true of both the pulses, they must both arrive at this point together. At the instant of their meeting, things are in the same condition as when the pulses were originated, and the movements just described will again take place. The period of a complete vibration of the string is therefore the time required for a pulse to travel over twice its length; that is,

$$\frac{1}{n}=\frac{2l}{v}=2l\sqrt{\frac{m}{t}};$$

$$\text{or } n=\frac{1}{2l}\sqrt{\frac{t}{m}}; \qquad (2)$$

l denoting the length of the string between its points of attachment, and n the number of vibrations per second.

This formula involves the following laws:—

1. When the length of the vibrating portion of the string is altered, without change of tension, the frequency of vibration varies inversely as the length.

2. If the tension be altered, without change of length in the vibrating portion, the frequency of vibration varies as the square root of the tension.

3. Strings of the same length, stretched with the same forces, have frequencies of vibration which are inversely as the square roots of their masses (or weights).

4. Strings of the same length and density, but of different thicknesses, will vibrate in the same time, if they are stretched with forces proportional to their sectional areas.

All these laws are illustrated (qualitatively, if not quantitatively) by the strings of a violin.

The first is illustrated by the fingering, the pitch being raised as the portion of string between the finger and the bridge is shortened.

The second is illustrated by the mode of tuning, which consists in tightening the string if its pitch is to be raised, or slackening the string if it is to be lowered.

The third law is illustrated by the construction of the bass string, which is wrapped round with metal wire, for the purpose of adding to its mass, and thus attaining slow vibration without undue slackness. The tension of this string is in fact greater than that of the string next it, though the latter vibrates more rapidly in the ratio of 3 to 2.

The fourth law is indirectly illustrated by the sizes of the first three strings. The treble string is the smallest, and is nevertheless stretched with much greater force than any of the others. The third string is the thickest, and is stretched with less force than any of the others. The increased thickness is necessary in order to give sufficient power in spite of the slackness of the string.

902. Experimental Illustration: Sonometer.—For the quantitative illustration of these laws, the instrument called the sonometer, represented in Fig. 618, is commonly employed. It consists essen-

tially of a string or wire stretched over a sounding-box by means of a weight. One end of the string is secured to a fixed point at one end of the sounding-box. The other end passes over a pulley, and carries weights which can be altered at pleasure. Near the two ends of the box are two fixed bridges, over which the cord passes. There is also a movable bridge, which can be employed for altering the length of the vibrating portion.

To verify the law of lengths, the whole length between the fixed bridges is made to vibrate, either by plucking or bowing; the mov-

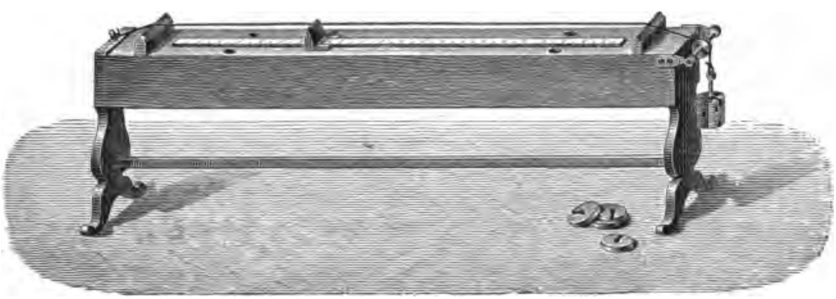


Fig. 618.—Sonometer.

able bridge is then introduced exactly in the middle, and one of the halves is made to vibrate; the note thus obtained will be found to be the upper octave of the first. The frequency of vibration is therefore doubled. By making two-thirds of the whole length vibrate, a note will be obtained which will be recognized as the fifth of the fundamental note, its vibration-frequency being therefore greater in the ratio $\frac{3}{2}$. To obtain the notes of the gamut, we commence with the string as a whole, and then employ portions of its length represented by the fractions $\frac{8}{9}$, $\frac{4}{5}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{8}{15}$, $\frac{1}{2}$.

To verify the law independently of all knowledge of musical intervals, a light style may be attached to the cord, and caused to trace its vibrations on the vibroscope. This mode of proof is also more general, inasmuch as it can be applied to ratios which do not correspond to any recognized musical interval.

To verify the law of tensions, we must change the weight. It will be found that, to produce a rise of an octave in pitch, the weight must be increased fourfold.

To verify the third and fourth laws, two strings must be employed, their masses having first been determined by weighing them.

If the strings are thick, and especially if they are thick steel wires, their flexural rigidity has a sensible effect in making the vibrations quicker than they would be if the tension acted alone.

903. Harmonics.—Any person of ordinary musical ear may easily, by a little exercise of attention, detect in any note of a piano the presence of its upper octave, and of another note a fifth higher than this; these being the notes which correspond to frequencies of vibration double and triple that of the fundamental note. A highly trained ear can detect the presence of other notes, corresponding to still higher multiples of the fundamental frequency of vibration. Such notes are called *harmonics*.

When the vibration-frequency of one note is an exact multiple of that of another note, the former note is called a harmonic of the latter. The notes of all stringed instruments contain numerous harmonics blended with the fundamental tones. Bells and vibrating plates have higher tones mingled with the fundamental tone; but these higher tones are not harmonics in the sense in which we use the word.

A violin string sometimes fails to yield its fundamental note, and gives the octave or some other harmonic instead. This result can be brought about at pleasure, by lightly touching the string at a properly-selected point in its length, while the bow is applied in the usual way. If touched at the middle point of its length, it gives the octave. If touched at one-third of its length from either end, it gives the fifth above the octave. The law is, that if touched at $\frac{1}{n}$ of its length[1] from either end, it yields the harmonic whose vibration-frequency is n times that of the fundamental tone. The string in these cases divides itself into a number of equal vibrating-segments, as shown in Fig. 619.

The division into segments is often distinctly *visible* when the string of a sonometer is strongly bowed, and its existence can be verified, when less evident, by putting paper riders on different parts of the string. These (as shown in the figure) will be thrown off by the vibrations of the string, unless they are placed accurately at the nodal points, in which case they will retain their seats. If two strings tuned to unison are stretched on the same sonometer, the vibration of the one induces similar vibrations in the other; and the experiment of the riders may be varied, in a very instructive way,

[1] Or at $\frac{m}{n}$ of its length, if m be prime to n.

by bowing one string and placing the riders on the other. This is an instance of a general principle of great importance—that a vibrating body communicates its vibrations to other bodies which are capable of vibrating in unison with it. The propagation of a sound may indeed be regarded as one grand vibration in unison; but, besides the general waves of *propagation,* there are waves of *re-*

Fig. 619.—Production of a Harmonic.

inforcement, due to the synchronous vibrations of limited portions of the transmitting medium. This is the principle of resonance.

904. Resonance.—By applying to a pendulum originally at rest a series of very feeble impulses, at intervals precisely equal to its natural time of vibration, we shall cause it to swing through an arc of considerable magnitude.

The same principle applies to a body capable of executing vibrations under the influence of its own elasticity. A series of impulses keeping time with its own natural period may set it in powerful vibration, though any one of them singly would have no appreciable effect.

Some bodies, such as strings and confined portions of air, have definite periods in which they can vibrate freely when once started;

and when a note corresponding to one of these periods is sounded in their neighbourhood, they readily take it up and emit a note of the same pitch themselves.

Other bodies, especially thin pieces of dry straight-grained deal, such as are employed for the faces of violins and the sounding-boards of pianos, are capable of vibrating, more or less freely, in any period lying between certain wide limits. They are accordingly set in vibration by all the notes of their respective instruments; and by the large surface with which they act upon the air, they contribute in a very high degree to increase the sonorous effect. All stringed instruments are constructed on this principle; and their quality mainly depends on the greater or less readiness with which they respond to the vibrations of the strings.

All such methods of reinforcing a sound must be included under *resonance;* but the word is often more particularly applied to the reinforcement produced by masses of air.

905. Longitudinal Vibrations of Strings.—Strings or wires may also be made to vibrate *longitudinally*, by rubbing them, in the direction of their length, with a bow or a piece of chamois leather covered with rosin. The sounds thus obtained are of much higher pitch than those produced by transverse vibration.

In the case of the fundamental note, each of the two halves A C, C B (Fig. 620), is alternately extended and compressed, one being

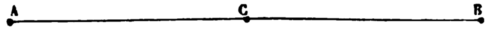


Fig. 620.—Longitudinal Vibration. First Tone.

extended while the other is compressed. At the middle point C there is no extension or compression, but there is greater amplitude of movement than at any other point. The amplitudes diminish in passing from C towards either end, and vanish at the ends, which are therefore nodes. The extensions and compressions, on the other hand, increase as we travel from the middle towards either end, and obtain their greatest values at the ends.

But the string may also divide itself into any number of separately-vibrating segments, just as in the case of transverse vibrations. Fig. 621 represents the motions which occur when there are three such segments, separated by two nodes D, E. The upper portion of the figure is true for one-half of the period of vibration, and the lower portion for the remaining half.

The frequency of vibration, for longitudinal as well as for transverse vibrations, varies inversely as the length of the vibrating string, or segment of string. We shall return to this subject in § 916.

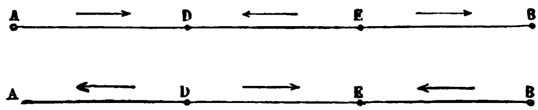


Fig. 621.—Longitudinal Vibration. Third Tone.

906. Stringed Instruments.—Only the transversal vibrations of strings are employed in music. In the violin and violoncello there are four strings, each being tuned a fifth above the next below it; and intermediate notes are obtained by fingering, the portion of string between the finger and the bridge being the only part that is free to vibrate. The bridge and sounding-post serve to transmit the vibrations of the strings to the body of the instrument. In the piano there is also a bridge, which is attached to the sounding-board, and communicates to it the vibrations of the wires.

907. Transversal Vibrations of Rigid Bodies: Rods, Plates, Bells.—We shall not enter into detail respecting the laws of the transverse vibrations of rigid bodies. The relations of their overtones to their fundamental tones are usually of an extremely complex character, and this fact is closely connected with the unmusical or only semi-musical character of the sounds emitted.

When one face of the body is horizontal, the division into separate vibrating segments can be rendered visible by a method devised by Chladni, namely, by strewing sand on this face. During the vibration, the sand, as it is tossed about, works its way to certain definite lines, where it comes nearly to rest. These nodal lines must be regarded as the intersections of internal nodal surfaces with the surface on which the sand is strewed, each nodal surface being the boundary between parts of the body which have opposite motions.

The figures composed by these nodal lines are often very beautiful, and quite startling in the suddenness of their production. Chladni and Savart published the forms of a great number. A complete theoretical explanation of them would probably transcend the powers of the greatest mathematicians.

Bells and bell-glasses vibrate in segments, which are never less than four in number, and are separated by nodal lines meeting in the middle of the crown. They are well shown by putting water in a

bell-glass, and bowing its edge. The surface of the water will immediately be covered with ripples, one set of ripples proceeding from each of the vibrating segments. The division into any possible number of segments may be effected by pressing the glass with the fingers in the places where a pair of consecutive nodes ought to be formed, while the bow is applied to the middle of one of the segments. The greater the number of segments the higher will be the note emitted.

908. Tuning-fork.—Steel rods, on account of their comparative freedom from change, are well suited for standards of pitch. The tuning-fork, which is especially used for this purpose, consists essentially of a steel rod bent double, and attached to a handle of the same material at its centre. Besides the fundamental tone, it is capable of yielding two or three overtones, which are very much higher in pitch; but these are never used for musical purposes. If the fork is held by the handle while vibrating, its motion continues for a long time, but the sound emitted is too faint to be heard except by holding the ear near it. When the handle is pressed against a table, the latter acts as a sounding-board, and communicates the vibrations to the air, but it also causes the fork to come much more speedily to rest. For the purposes of the lecture-room the fork is often mounted on a sounding-box (Fig. 622), which should be separated from the table by two pieces of india-rubber tubing. The box can then vibrate freely in unison with the fork, and the sound is both loud and lasting. The vibrations are usually excited either by bowing the fork or by drawing a piece of wood between its prongs.

Fig. 622.—Fork on Sounding-box.

The pitch of a tuning-fork varies slightly with temperature, becoming lower as the temperature rises. This effect is due in some trifling degree to expansion, but much more to the diminution of elastic force.

909. Law of Linear Dimensions.—The following law is of very wide application, being applicable alike to solid, liquid, and gaseous bodies: —*When two bodies differing in size, but in other respects similar and similarly circumstanced, vibrate in the same mode, their vibration-periods are directly as their linear dimensions.* Their vibra-

tion-frequencies are consequently in the inverse ratio of their linear dimensions.

In applying the law to the transverse vibrations of strings, it is to be understood that the stretching force per unit of sectional area is constant. In this case the velocity of a pulse (§ 901) is constant, and the period of vibration, being the time required for a pulse to travel over twice the length of the string, is therefore directly as the length.

910. Organ-pipes.—In organs, and wind-instruments generally, the sonorous body is a column of air confined in a tube. To set this air

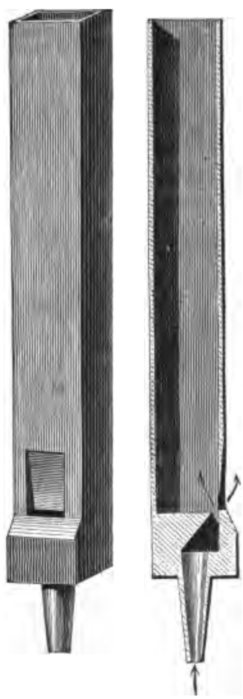


Fig. 623.—Block Pipe.

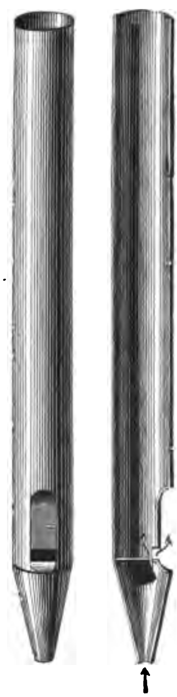


Fig. 624.—Flue Pipe.

in vibration some kind of mouth-piece must be employed. That which is most extensively used in organs is called the *flute mouth-piece,*[1] and is represented, in conjunction with the pipe to which it is attached, in Figs. 623, 624. It closely resembles the mouth-piece of

[1] This is not the trade name. English organ-builders have no generic name for this mouth-piece.

an ordinary whistle. The air from the bellows arrives through the conical tube at the lower end, and, escaping through a narrow slit, grazes the edge of a wedge placed opposite. A rushing noise is thus produced, which contains, among its constituents, the note to which the column of air in the pipe is capable of resounding; and as soon as this resonance occurs, the pipe speaks. Fig. 623 represents a wooden and Fig. 624 a metal organ-pipe, both of them being furnished with flute mouth-pieces. The two arrows in the sections are intended to suggest the two courses which the wind may take as it issues from the slit, one of which it actually selects to the exclusion of the other.

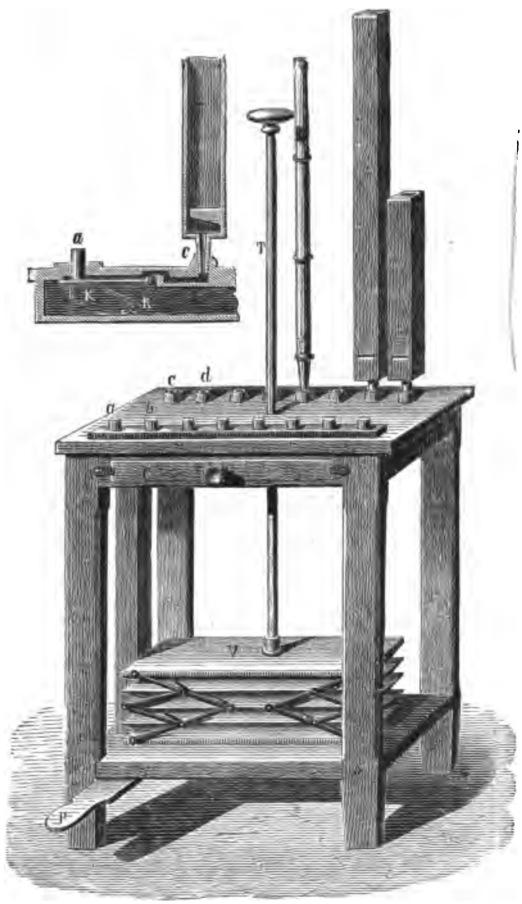


Fig. 625.—Experimental Organ.

The arrangements for admitting the wind to the pipes by putting down the keys are shown in Fig. 625. The bellows V are worked by the treadle P. The force of the blast can be increased by weighting the top of the bellows, or by pressing on the rod T. The air passes up from the bellows, through a large tube shown at one end, into a reservoir C, called the wind-chest. In the top of the wind-chest there are numerous openings *c*, *d*, &c., in which the tubes are to be fixed. The sectional drawing in the upper part of the figure shows the internal communications. A plate K, pressed up by a spring R, cuts off the tube *c* from the wind-chest, until the pin *a*

is depressed. The putting down of this pin lowers the plate, and admits the wind. This description only applies to the experimental organs which are constructed for lecture illustration. In real organs the pressure of the wind in the bellows is constant; and as this pressure would be too great for most of the pipes, the several apertures of admission are partially plugged, to diminish the force of the blast.

911. The Air is the Sonorous Body.—It is easily shown that the sound emitted by an organ-pipe depends, mainly at least, on the dimensions of the inclosed column of air, and not on the thickness or material of the pipe itself. For let three pipes, one of wood, one of copper, and the other of thick card, all of the same internal dimensions, be fixed on the wind-chest. On making them speak, it will be found that the three sounds have exactly the same pitch, and but slight difference in character. If, however, the sides of the tube are *excessively* thin, their yielding has a sensible influence, and the pitch of the sound is modified.

912. Law of Linear Dimensions.—The law of linear dimensions, stated in § 909 as applying to the vibrations of similar solid bodies, applies to gases also. Let two box-shaped pipes (Fig. 626) of precisely similar form, and having their linear dimensions in the ratio of 2 : 1, be fixed on the wind-chest; it will be found, on making them speak, that the note of the small one is an octave higher than the other;—showing double frequency of vibration.

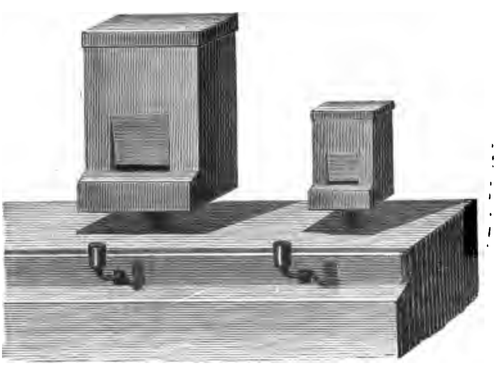


Fig. 626.—Law of Linear Dimensions.

913. Bernoulli's Laws.—The law just stated applies to the comparison of similar tubes of any shape whatever. When the length of a tube is a large multiple of its diameter, the note emitted is nearly independent of the diameter, and depends almost entirely on the length. The relations between the fundamental note of such a tube and its overtones were discovered by Daniel Bernoulli, and are as follows:—

I. *Overtones of Open Pipes.*—Let the pipe B (Fig. 627), which is

open at the upper end, be fixed on the wind-chest; let the corresponding key be put down, and the wind gradually turned on, by means of the cock below the mouth-piece. The first note heard will be feeble and deep; it is the fundamental note of the pipe. As the wind is gradually turned full on, and increasing pressure afterwards applied to the bellows, a series of notes will be heard, each higher than its predecessor. These are the overtones of the pipe. They are the harmonics of the fundamental note; that is to say, if 1 denote the frequency of vibration for the fundamental tone, the frequencies of vibration for the overtones will be approximately 2, 3, 4, 5 . . . respectively.

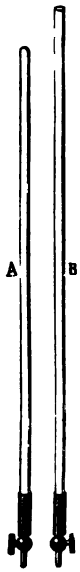


Fig. 627. Tubes for Overtones.

II. *Overtones of Stopped Pipes.*—If the same experiment be tried with the pipe A, which is closed at its upper end; the overtones will form the series of odd harmonics of the fundamental note, all the even harmonics being absent; in other words, the frequencies of vibration of the fundamental tone and overtones will be approximately represented by the series of odd numbers 1, 3, 5, 7 . . .

It will also be found, that if both pipes are of the same length, the fundamental note of the stopped pipe is an octave lower than that of the open pipe.

914. Mode of Production of Overtones.—In the production of the overtones, the column of air in a pipe divides itself into vibrating segments, separated by nodal cross-sections. At equal distances on opposite sides of a node, the particles of air have always equal and opposite velocities, so that the air at the node is always subjected to equal forces in opposite directions, and thus remains unmoved by their action. The portion of air constituting a vibrating segment, sways alternately in opposite directions, and as the movements in two consecutive segments are opposite, two consecutive nodes are always in opposite conditions as regards compression and extension. The middle of a vibrating segment is the place where the amplitude of vibration is greatest, and the variation of density least. It may be called an *antinode.* The distance from one node to the next is half a wave-length, and the distance from a node to an antinode is a quarter of a wave-length. Both ends of an open pipe, and the end next the mouth-piece of a stopped pipe, are antinodes, being preserved from changes of density by their free communication with

the external air. At the closed end of a stopped pipe there must always be a node.

The swaying to and fro of the internodal portions of air between fixed nodal planes, is an example of *stationary undulation;* and the vibration of a musical string is another example. A stationary undulation may always be analysed into two component undulations equal and similar to one another, and travelling in opposite directions, their common wave-length being double of the distance from node to node (§ 887). These undulations are constantly undergoing reflection from the ends of the pipe or string, and, in the case of pipes, the reflection is opposite in kind according as it takes place from a closed or an open end. In the former case a condensation propagated towards the end is reflected as a condensation, the forward-moving particles being compelled to recoil by the resistance which they there encounter; and a rarefaction is, in like manner, reflected as a rarefaction. On the other hand, when a condensation arrives at an open end, the sudden opportunity for expansion which is afforded causes an outward movement in excess of that which would suffice for equilibrium of pressure, and a rarefaction is thus produced which is propagated back through the tube. A condensation is thus reflected as a rarefaction; and a rarefaction is, in like manner, reflected as a condensation.

The period of vibration of the fundamental note of a stopped pipe is the time required for propagating a pulse through four times the length of the pipe. For let a condensation be suddenly produced at the lower end by the action of the vibrating lip. It will be propagated to the closed end and reflected back, thus travelling over twice the length of the pipe. On arriving at the aperture where the lip is situated, it is reflected as a rarefaction. This rarefaction travels to the closed end and back, as the condensation did before it, and is then reflected from the aperture as a condensation. Things are now in their initial condition, and one complete vibration has been performed. The period of the movements of the lip is determined by the arrival of these alternate condensations and rarefactions; and the lip, in its turn, serves to divert a portion of the energy of the blast, and employ it in maintaining the energy of the vibrating column.

The wave-length of the fundamental note of a stopped pipe is thus four times the length of the pipe.

In an open pipe, a condensation, starting from the mouth-piece, is reflected from the other end as a rarefaction. This rarefaction, on

reaching the mouth-piece, is reflected as a condensation; and things are thus in their initial state after the length of the pipe has been traversed twice. The period of vibration of the fundamental note is accordingly the time of travelling over twice the length of the pipe; and its wave-length is twice the length of the pipe. In every case of longitudinal vibration, if the reflection is alike at both ends, the wave-length of the fundamental tone is twice the distance between the ends.

915. Explanation of Bernoulli's Laws.—In investigating the theoretical relations between the fundamental tone and overtones for a pipe of either kind, it is convenient to bear in mind that the distance from an open end to the nearest node is a quarter of a wave-length of the note emitted.

In the case of the open pipe the first or fundamental tone requires one node, which is at the middle of the length. The second tone requires two nodes, with half a wave-length between them, while each of them is a quarter of a wave-length from the nearest end. A quarter wave-length has thus only half the length which it had for the fundamental tone, and the frequency of vibration is therefore doubled.

The third tone requires three nodes, and the distance from either end to the nearest node is $\frac{1}{6}$ of the length of the pipe, instead of $\frac{1}{2}$ the length as in the case of the first tone. The wave-length is thus divided by 3, and the frequency of vibration is increased threefold. We can evidently account in this way for the production of the complete series of harmonics of the fundamental note.

In the case of the stopped pipe, the mouth-piece is always distant a quarter wave-length from the nearest node, and this must be distant an even number of quarter wave-lengths from the stopped end, which is itself a node.

For the fundamental tone, a quarter wave-length is the whole length of the pipe.

For the second tone, there is one node besides that at the closed end, and its distance from the open end is $\frac{1}{3}$ of the length of the pipe.

For the third tone, there are two nodes besides that at the closed end. The distance from the open end to the nearest node is therefore $\frac{1}{5}$ of the length of the pipe.

The wave-lengths of the successive tones, beginning with the fundamental, are therefore as 1, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$ . . . , and their vibration-frequencies are as 1, 3, 5, 7 . . .

Also, since the wave-length of the fundamental tone is four times the length of the pipe if stopped, and only twice its length if open, it is obvious that the wave-length is halved, and the frequency of vibration doubled, by unstopping the pipe.

No change of pitch, or only very slight change, will be produced by inserting a solid partition at a node, or by putting an antinode in free communication with the external air. These principles can be illustrated by means of the jointed pipe represented in Fig. 628.

Fig. 628. Jointed Pipe.

916. Application to Rods and Strings.—The same laws which apply to open organ-pipes, also apply to the longitudinal vibrations of rods free at both ends, and to both the longitudinal and transverse vibrations of strings. In all these cases the overtones form the complete series of harmonics of the first or fundamental tone, and the period of vibration for this first tone is the time occupied by a pulse in travelling over twice the length of the given rod or string. In the case of longitudinal vibrations the velocity of a pulse is $\sqrt{\frac{M}{D}}$, M denoting the value of Young's modulus for the rod or string, and D its density. This is identical with the velocity of sound through the rod or string, and is independent of its tension. In the case of transverse pulses in a string (regarded as perfectly flexible), the formula for the velocity of transmission (1) § 901, may be written $\sqrt{\frac{F}{D}}$, F denoting the stretching force per unit of sectional area. The ratio of the latter velocity to the former is $\sqrt{\frac{F}{M}}$, which is always a small fraction, since $\frac{F}{M}$ expresses the fraction of itself by which the string is lengthened by the force F.

If a rod, free at both ends, is made to vibrate longitudinally, its nodes and antinodes will be distributed exactly in the same way as those of an open organ-pipe. The experiment can be performed by holding the rod at a node, and rubbing it with rosined chamois leather.

917. Application to Measurement of Velocity in Gases.—Let v denote the velocity of sound in a particular gas, in feet per second, λ the wave-length of a particular note in this gas in feet, and n the frequency of vibration for this note, that is the number of vibrations

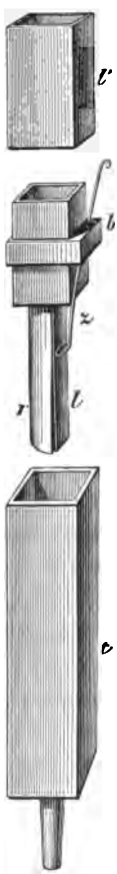
per second which produce it. Then λ is the distance travelled in $\frac{1}{n}$ of a second, and the distance travelled in a second is

$$v = n\lambda.$$

For the same note, n is constant for all media whatever, and v varies directly as λ. The velocities of sound in two gases may thus be compared by observing the lengths of vibrating columns of the two gases which give the same note; or if columns of equal length be employed, the velocities will be directly as the frequencies of vibration, which are determined by observing the pitch of the notes emitted.

By these methods, Dulong, and more recently Wertheim, have determined the velocity of sound in several different gases. The following are Wertheim's results, in metres per second, the gases being supposed to be at 0° C.

Air,	331	Carbonic acid,	262
Oxygen,	317	Nitrous oxide,	262
Hydrogen,	1269	Olefiant gas,	314
Carbonic oxide,	337		



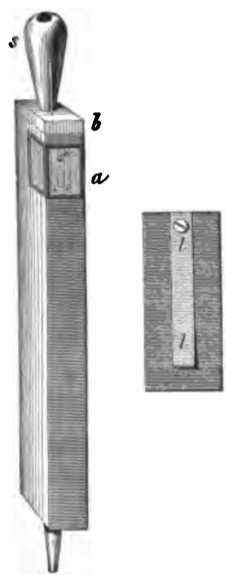


Fig. 629.—Reed Pipe. Fig. 630.—Free Reed.

The same principle is applicable to liquids and solids; and it was by means of the longitudinal vibrations of rods that the velocities given in § 880 were ascertained.

918. Reed-pipes.—Instead of the flute mouth-piece above described, organ-pipes are often furnished with what is called a *reed*. A reed contains an elastic plate l (Figs. 629, 630) called the *tongue*, which, by its vibrations, alternately opens and closes or nearly closes an aperture through which the wind passes. In Fig. 629, the air from the bellows enters first

the lower part t of the pipe, and thence (when permitted by the tongue) passes through the channel[1] r into the upper part t'. The stiff wire z, movable with considerable friction through the hole b, limits the vibrating portion of the tongue, and is employed for tuning. Reed-pipes are often terminated above by a trumpet-shaped expansion.

A *striking reed* (Fig. 629) is one whose tongue closes the aperture by covering it. The tongue should be so shaped as not to strike along its whole length at once, but to roll itself down over the aperture. In the *free reed* (Fig. 630) the tongue can pass completely through.

The striking reed is generally preferred in organs, its peculiar character rendering it very effective by way of contrast. It is always used for the *trumpet* stop. Reed-pipes can be very strongly blown without breaking into overtones. Their pitch, however, if they are of the striking kind, is not independent of the pressure of the wind, but gradually rises as the pressure increases. Free reeds, which are used for harmoniums, accordions, and concertinas, do not change in pitch with change of pressure.

Elevation of temperature sharpens pipes with flute mouth-pieces, and flattens reed-pipes. The sharpening is due to the increased velocity of sound in hot air. The flattening is due to the diminished elasticity of the metal tongue. It is thus proved that the pitch of a reed-pipe is not always that due to the free vibration of the inclosed air, but may be modified by the action of the tongue.

919. Wind-instruments.—In all wind-instruments, the sound is originated by one of the two methods just described. With the flute-pipe must be classed the flute, the flageolet, and the Pandean-pipes. The clarionet, hautboy, and bassoon have reed mouth-pieces, the vibrating tongue being a piece of reed or cane. In the bugle, trumpet, and French-horn, which are mere tubes without keys, the lips of the performer act as the reed-tongue, and the notes produced are approximately the natural overtones. These, when of high order, are so near together, that a gamut can be formed by properly selecting from among them.

The fingering of the flute and clarionet, has the effect sometimes of altering the effective length of the vibrating column of air, and sometimes of determining the production of overtones. In the

[1] The piece r, which is approximately a half cylinder, is called the *reed* by organ-builders.

trombone and cornet-à-piston, the length of the vibrating column of air is altered. The harmonium, accordion, and concertina are reed instruments, the reeds employed being always of the free kind.

920. Manometric Flames.—Koenig, of Paris, constructs several forms of apparatus, in which the variations of pressure produced by vibrations of air in a pipe are rendered evident to the eye by their effect upon flames. One of these is represented in Fig. 631. Three small gas-burners are fixed at definite points in the side of a pipe, as represented in the figure. When the pipe gives its second tone, the central flame is at an antinode and remains unaffected, while the other two, being at nodes, are agitated or blown out. When it gives its first tone, the central flame, which is now at a node, is more powerfully affected than the others. The gas which supplies these burners is separated from the air in the pipe only by a thin membrane. When the pipe is made to speak, the flame at the node is violently agitated, in consequence of the changes of pressure on the back of the membrane, while those at the ventral points are scarcely affected. The agitation of the flame is a true vibration; and, when examined by the aid of a revolving mirror, presents the appearance of tongues of flame alternating with nearly dark spaces. If two pipes, one an octave higher than the other, are connected with the same gas flame, or with two gas flames which can be viewed in the same mirror, the tongues of flame corresponding to the upper octave are seen to be twice as numerous as the others.

Fig. 631.—Manometric Flames.

CHAPTER LXV.

ANALYSIS OF VIBRATIONS. CONSTITUTION OF SOUNDS.

921. Optical Examination of Sonorous Vibrations.—Sound is a special sensation belonging to the sense of hearing; but the vibrations which are its physical cause often manifest themselves to other senses. For instance, we can often feel the tremors of a sonorous body by touching it; we see the movements of the sand on a vibrating plate, the curve traced by the style of a vibroscope, &c. The aid which one sense can thus furnish in what seems the peculiar province of another is extremely interesting. M. Lissajous has devised a very beautiful optical method of examining sonorous vibrations, which we will briefly describe.

922. Lissajous' Experiment.—Suppose we introduce into a dark room (Fig. 632) a beam of solar rays, which, after passing through a lens L, is reflected, first, from a small mirror fixed on one of the branches of a tuning-fork D, and then from a second mirror M, which throws it on a screen E; we can thus, by proper adjustments, form upon the screen a sharp and bright image of the sun, which will appear as a small spot of light. As long as the apparatus remains at rest, we shall not observe any movement of the image; but if the tuning-fork vibrates, the image will move rapidly up and down along the line I, I′, producing, in consequence of the persistence of impressions, the appearance of a vertical line of light. If the tuning-fork remains at rest, but the mirror M is rotated through a small angle about a vertical axis, the image will move horizontally. Consequently, if both these motions take place simultaneousely, the spot of light will trace out on the screen a sinuous line, as represented in the figure, each S-shaped portion corresponding to one vibration of the tuning-fork.

Now, let the mirror M be replaced by a small mirror attached to

a second tuning-fork, which vibrates in a horizontal plane, as in

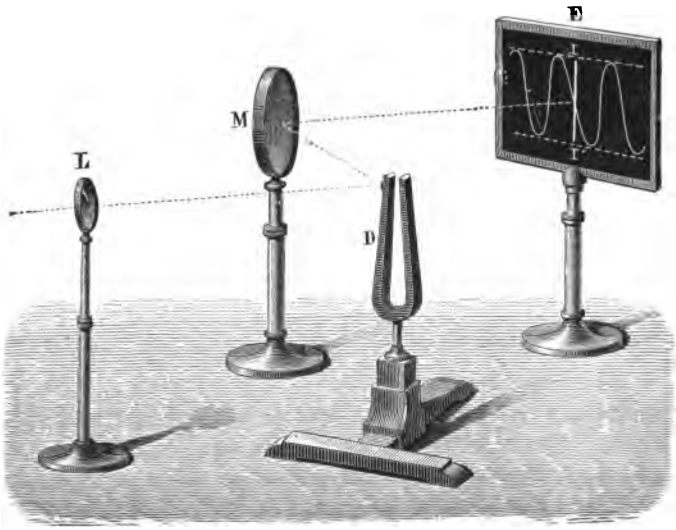


Fig. 632.—Principle of Lissajous' Experiment.

Fig. 633. If this fork vibrates alone, the image will move to and

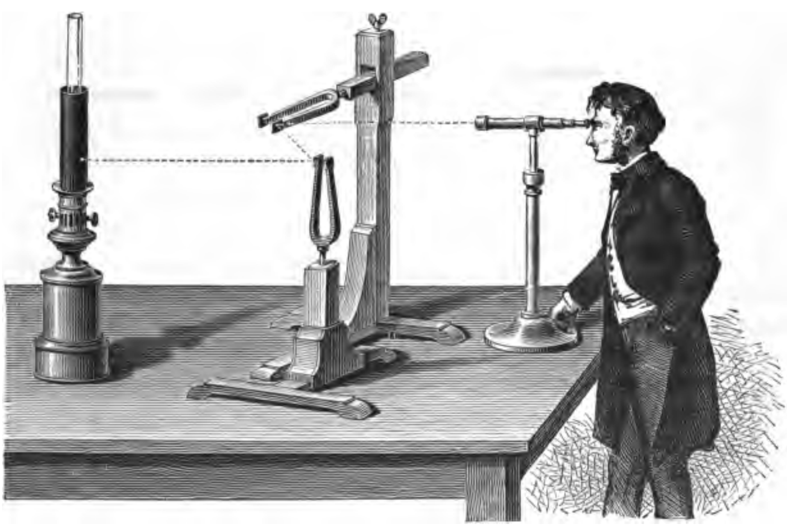


Fig. 633.—Lissajous' Experiment.

fro horizontally, presenting the appearance of a horizontal line of

light, which gradually shortens as the vibrations die away. If both forks vibrate simultaneously, the spot of light will rise and fall according to the movements of the first fork, and will travel left and right according to the movements of the second fork. The curve actually described, as the resultant of these two component motions, is often extremely beautiful. Some varieties of it are represented in Fig. 634.

Instead of throwing the curves on a screen, we may see them by looking into the second mirror, either with a telescope, as in Fig. 633,

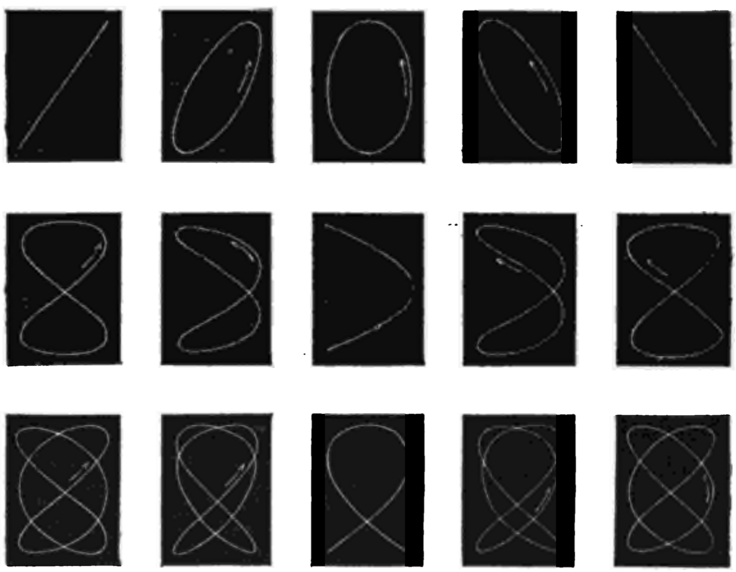


Fig. 634.—Lissajous' Figures, Unison, Octave, and Fifth.

or with the naked eye. In this form of the experiment, a lamp surrounded by an opaque cylinder, pierced with a small hole just opposite the flame, as represented in the figure, is a very convenient source of light.

The movement of the image depends almost entirely on the angular movements of the mirrors, not on their movements of translation; but the distinction is of no importance, for, in the case of such small movements, the linear and angular changes may be regarded as strictly proportional.

Either fork vibrating alone would cause the image to execute *simple harmonic motion* (§§ 109–111), or, as it may conveniently

be called, *simple vibration;* so that the movement actually executed will be the resultant of two simple harmonic motions in directions perpendicular to each other.

Suppose the two forks to be in unison. Then the two simple harmonic motions will have the same period, and the path described will always be some kind of ellipse,[1] the circle and straight line being included as particular cases. It will be a straight line if both forks pass through their positions of equilibrium at the same instant. In order that it may be a circle, the amplitudes of the two simple harmonic motions must be equal, and one fork must be in a position of maximum displacement when the other is in the position of equilibrium.

If the unison were rigorous, the curve once obtained would remain unchanged, except in so far as its breadth and height became reduced by the dying away of the vibrations. But this perfect unison is never attained in practice, and the eye detects changes depending on differences of pitch too minute to be perceived by the ear. These changes are illustrated by the upper row of forms in Fig. 634, commencing, say, with the sloping straight line at the left hand, which gradually opens out into an ellipse, and afterwards contracts into a straight line, sloping the opposite way. It then retraces its steps, the motion being now in opposition to the arrows in the figure, and then repeats the same changes.

If the interval between the two forks is an octave, we shall obtain the curves represented in the second row;[2] if the interval is a fifth, we shall obtain the curves in the lowest row. In each case the order of the changes will be understood by proceeding from left to right,

[1] Employing horizontal and vertical co-ordinates, and denoting the amplitudes by a and b, we have, in the case of unison, $\frac{x}{a} = \sin\theta$, $\frac{y}{b} = \sin(\theta + \beta)$, where β denotes the difference of phase, and θ is an angle varying directly as the time. Eliminating θ, we obtain the equation to an ellipse, whose form and dimensions depend upon the given quantities, a, b, β.

[2] The middle curve in this row is a parabola, and corresponds to the elimination of θ between the equations $\frac{x}{a} = \cos 2\theta$, $\frac{y}{a} = \cos\theta$. The coefficient 2 indicates the double frequency of horizontal as compared with vertical vibrations.

The general equations to Lissajous' figures are $\frac{x}{a} = \sin m\theta$, $\frac{y}{b} = \sin(n\theta + \beta)$, where m and n are proportional to the frequencies of horizontal and vertical vibrations. The gradual changes from one figure to another depend on the gradual change of β, and all the figures can be inscribed in a rectangle, whose length and breadth are $2a$ and $2b$.

and then back again; but the curves obtained in returning will be inverted.

923. Optical Tuning.—By the aid of these principles, tuning-forks can be compared with a standard fork with much greater precision than would be attainable by ear. Fig. 635 represents a convenient

Fig. 635.—Optical Comparison of Tuning-forks.

arrangement for this purpose. A lens *f* is attached to one of the prongs of a standard fork, which vibrates in a horizontal plane; and above it is fixed an eye-piece *g*, the combination of the two being equivalent to a microscope. The fork to be compared is placed upright beneath, and vibrates in a vertical plane, the end of one prong being in the focus of the microscope. A bright point *m*, produced by making a little scratch on the end of the prong with a diamond, is observed through the microscope, and is illuminated, if necessary, by converging a beam of light upon it through the lens *c*. When the forks are set vibrating, the bright point is seen as a luminous ellipse, whose permanence of form is a test of the closeness of the unison. The ellipse will go through a complete cycle of changes in the time required for one fork to gain a complete vibration on the other.

924. Other Modes of producing Lissajous' Figures.—An arrangement devised in 1844 by Professor Blackburn, of Glasgow, then a student at Cambridge, affords a very easy mode of obtaining, by a slow motion, the same series of curves which, in the above arrangements, are obtained by a motion too quick for the eye to follow. A cord A B C (Fig. 636) is fastened at A and C, leaving more or less

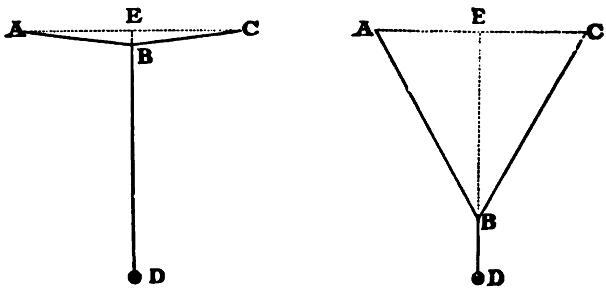


Fig. 636.—Blackburn's Pendulum.

slack, according to the curves which it is desired to obtain; and to any intermediate point B of the cord another string is tied, carrying at its lower end a heavy body D to serve as pendulum-bob.

If, when the system is in equilibrium, the bob is drawn aside in the plane of A B C and let go, it will execute vibrations in that plane, the point B remaining stationary, so that the length of the pendulum is B D. If, on the other hand, it be drawn aside in a plane perpendicular to the plane A B C, it will vibrate in this perpendicular plane, carrying the whole of the string with it in its motion, so that the length of the pendulum is the distance of the bob from the point E, in which the straight line A C is cut by D B produced. The frequencies of vibration in the two cases will be inversely as the square roots of the pendulum-lengths B D, E D.

If the bob is drawn aside in any other direction, it will not vibrate in one plane, but will perform movements compounded of the two independent modes of vibration just described, and will thus describe curves identical with Lissajous'. If the ratio of E D to B D is nearly equal to unity, as in the left-hand figure, we shall have curves corresponding to approximate unison. If it be approximately 4, as in the right-hand figure, we shall obtain the curves of the octave. Traces of the curves can be obtained by employing for the bob a

vessel containing sand, which runs out through a funnel-shaped opening at the bottom.[1]

The curves can also be exhibited by fixing a straight elastic rod at one end, and causing the other end to vibrate transversely. This was the earliest known method of obtaining them. If the flexural rigidity of the rod is precisely the same for all transverse directions, the vibrations will be executed in one plane; but if there be any inequality in this respect, there will be two mutually perpendicular directions possessing the same properties as the two principal directions of vibration in Blackburn's pendulum. A small bright metal knob is usually fixed on the vibrating extremity to render its path visible. The instrument constructed for this mode of exhibiting the figures is called a *kaleidophone*. In its best form (devised by Professor Barrett) the upper and lower halves of the rod (which is vertical) are flat pieces of steel, with their planes at right angles, and a stand is provided for clamping the lower piece at any point of its length that may be desired, so as to obtain any required combination.

925. Character.—*Character* or *timbre*, which we have already defined in § 889, must of necessity depend on the *form* of the vibration of the aerial particles by which sound is transmitted, the word *form* being used in the metaphorical sense there explained, for in the literal sense the form is always a straight line. When the changes of density are represented by ordinates of a curve, as in Fig. 603, the form of this curve is what is meant by the form of vibration.

The subject of *timbre* has been very thoroughly investigated in recent years by Helmholtz; and the results at which he has arrived are now generally accepted as correct.

The first essential of a musical note is, that the aerial movements which constitute it shall be strictly *periodic;* that is to say, that each vibration shall be exactly like its successor, or at all events, that, if there be any deviation from strict periodicity, it shall be so gradual as not to produce sensible dissimilarity between several consecutive vibrations of the same particle.

There is scarcely any proposition more important in its application

[1] Mr. Hubert Airy has obtained very beautiful traces by attaching a glass pen to the bob (see *Nature*, Aug. 17 and Sept. 7, 1871), and in Tisley's *harmonograph* the same result is obtained by means of two pendulums, one of which moves the paper and the other the pen.

to modern physical investigations than the following mathematical theorem, which was discovered by Fourier:—*Any periodic vibration executed in one line can be definitely resolved into simple vibrations, of which one has the same frequency as the given vibration, and the others have frequencies* 2, 3, 4, 5 . . . *times as great,* no fractional multiples being admissible. The theorem may be briefly expressed by saying that *every periodic vibration consists of a fundamental simple vibration and its harmonics.*

We cannot but associate this mathematical law with the experimental fact, that a trained ear can detect the presence of harmonics in all but the very simplest musical notes. The analysis which Fourier's theorem indicates, appears to be actually performed by the auditory apparatus.

The *constitution* of a periodic vibration may be said to be known if we know the ratios of the amplitudes of the simple vibrations which compose it; and in like manner the constitution of a sound may be said to be known if we know the relative intensities of the different elementary tones which compose it.

Helmholtz infers from his experiments that the *character* of a musical note depends upon its *constitution* as thus defined; and that, while change of intensity in any of the components produces a modification of character, change of phase has no influence upon it whatever. Sir W. Thomson, in a paper "On Beats of Imperfect Harmonies,"[1] adduces strong evidence to show that change of phase has, in some cases at least, an influence on character.

The harmonics which are present in a note, usually find their origin in the vibrations of the musical instrument itself. In the case of stringed instruments, for example, along with the vibration of the string as a whole, a number of segmental vibrations are simultaneously going on. Fig. 637 represents curves obtained by the composition of the fundamental mode of vibration with another an octave higher. The broken lines indicate the forms which the string would assume if yielding only its fundamental note.[2] The continuous lines in the first and third figures are forms which a string may assume in its two positions of greatest displacement, when yielding the octave along with the fundamental, the time required for the

[1] *Proc. R. S. E.* 1878.

[2] The form of a string vibrating so as to give only one tone (whether fundamental or harmonic) is a curve of sines, all its ordinates increasing or diminishing in the same proportion, as the string moves.

string to pass from one of these positions to the other being the same as the time in which each of its two segments moves across and back

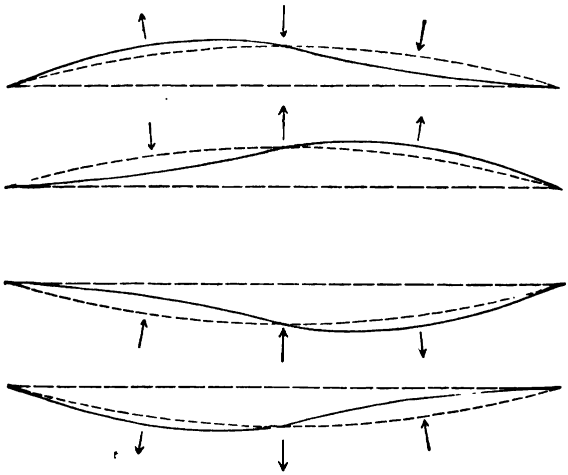


Fig. 637.—String giving first Two Tones.

again. The second and fourth figures must in like manner be taken together, as representing a pair of extreme positions. The number of harmonics thus yielded by a pianoforte wire is usually some four or five; and a still larger number are yielded by the strings of a violin.

The notes emitted from wide organ-pipes with flute mouth-pieces are very deficient in harmonics. This defect is remedied by combining with each of the larger pipes a series of smaller pipes,[1] each yielding one of its harmonics. An ordinary listener hears only one note, of the same pitch as the fundamental, but much richer in character than that which the fundamental pipe yields alone. A trained ear can recognize the individual harmonics in this case as in any other.

[1] The stops called *open diapason* and *stop diapason* (consisting respectively of open and stopped pipes), give the fundamental tone, almost free from harmonics. The stop absurdly called *principal* gives the second tone, that is the octave above the fundamental. The stops called *twelfth* and *fifteenth* give the third and fourth tones, which are a twelfth (octave + fifth), and a fifteenth (double octave) above the fundamental. The fifth, sixth, and eighth tones are combined to form the stop called *mixture*.

As many of our readers will be unacquainted with the structure of organs, it may be desirable to state that an organ contains a number of complete instruments, each consisting of several octaves of pipes. Each of these complete instruments is called a *stop*, and is brought into use at the pleasure of the organist by pulling out a slide, by means of a knob-handle, on which the name of the stop is marked. To throw it out of use, he pushes in the slide. A large number of stops are often in use at once.

It is important to remark, that though the presence of harmonic subdivisions in a vibrating body necessarily produces harmonics in the sound emitted, the converse cannot be asserted. Simple vibrations, executed by a vibrating body, produce vibrations *of the same frequency* as their own, in any medium to which they are transmitted, but not necessarily *simple* vibrations. If they produce compound vibrations, these, as we have seen (§ 925), must consist of a fundamental simple vibration and its harmonics.

926. **Helmholtz's Resonators.**—Helmholtz derived material aid in his researches from an instrument devised by himself, and called a *resonator* or *resonance globe* (Fig. 638). It is a hollow globe of thin

Fig. 638.—Resonator.

brass, with an opening at each end, the larger one serving for the admission of sound, while the smaller one is introduced into the ear. The inclosed mass of air has, like the column of air in an organ-pipe, a particular fundamental note of its own, depending upon its size; and whenever a note of this particular pitch is sounded in its neighbourhood, the inclosed air takes it up and intensifies it by resonance. In order to test the presence or absence of a particular harmonic in a given musical tone, a resonator, in unison with this harmonic, is applied to the ear, and if the resonator speaks it is known that the harmonic is present. These instruments are commonly constructed so as to form a series, whose notes correspond to the bass C of a man's voice, and its successive harmonics as far as the 10th or 12th.

Koenig has applied the principle of manometric flames to enable a large number of persons to witness the analysis of sounds by resonators. A series of 6 resonators, whose notes have frequencies proportional to 1, 2, 3, 4, 5, 6, are fixed on a stand (Fig. 639), and their smaller ends, instead of being applied to the ear, are connected each

with a separate manometric capsule, which acts on a gas jet. When the mirrors are turned, it is easy to see which of the flames vibrate while a sonorous body is passed in front of the resonators.

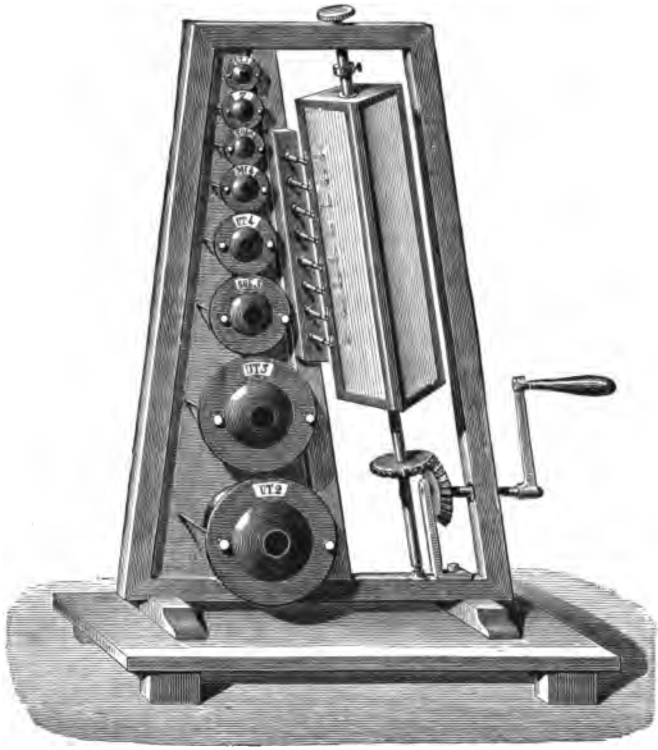


Fig. 639.—Analysis by Manometric Flames.

A simple tone, unaccompanied by harmonics, is dull and uninteresting, and, if of low pitch, is very destitute of penetrating quality.

Sounds composed of the first six elementary tones in fair proportion, are rich and sweet.

The higher harmonics, if sufficiently subdued, may also be present without sensible detriment to sweetness, and are useful as contributing to expression. When too loud, they render a sound harsh and grating; an effect which is easily explained by the discordant combinations which they form one with another; the 8th and 9th tones, for example, are at the same interval as the notes *Do* and *Re*.

927. Vowel Sounds.—The human voice is extremely rich in harmonics, as may be proved by applying the series of resonators to the

ear while the fundamental note is sung. The origin of the tones of the voice is in the vocal chords, which, when in use, form a diaphragm with a slit along its middle. The edges of this slit vibrate when air is forced through, and, by alternately opening and closing the passage, perform the part of a reed. The cavity of the mouth serves as a resonance chamber, and reinforces particular notes depending on the position of the organs of speech. It is by this resonance that the various vowel sounds are produced. The deepest pitch belongs to the vowel sound which is expressed in English by *oo* (as in *moon*), and the highest to *ee* (as in *screech*).

Willis in 1828[1] succeeded in producing the principal vowel sounds by a single reed fitted to various lengths of tube. Wheatstone, a few years later, made some advances in theory,[2] and constructed a machine by which nearly all articulate sounds could be imitated.

Excellent imitations of some of the vowel sounds can be obtained by placing Helmholtz's resonators, one at a time, on a free-reed pipe, the small end of the resonator being inserted in the hole at the top of the pipe.

The best determinations of the particular notes which are reinforced in the case of the several vowel sounds, have been made by Helmholtz, who employed several methods, but chiefly the two following:—

1. Holding resonators to the ear, while a particular vowel sound was loudly sung.

2. Holding vibrating tuning-forks in front of the mouth when in the proper position for pronouncing a given vowel; and observing which of them had their sounds reinforced by resonance.[3]

Helmholtz has verified his determinations synthetically. He employs a set of tuning-forks which are kept in vibration by the alternate making and unmaking of electro-magnets, the circuit being made and broken by the vibrations of one large fork of 64 vibrations per second. The notes of the other forks are the successive harmonics of this fundamental note. Each fork is accompanied by a

[1] *Cambridge Transactions*, vol. iii.

[2] *London and Westminster Review*, October, 1837.

[3] According to Koenig (*Comptes Rendus*, 1870) the notes of strongest resonance for the vowels *u*, *o*, *a*, *e*, *i*, as pronounced in North Germany, are the five successive octaves of B flat, commencing with that which corresponds to the space above the top line of the base clef. Willis, Helmholtz, and Koenig all agree as regards the note of the vowel *o*, which is very nearly that of a common A tuning-fork. They are also agreed respecting the note of *a* (as in *father*), which is an octave higher.

resonance-tube, which, when open, renders the note of the fork audible at a distance; and by means of a set of keys, like those of a piano, any of these tubes can be opened at pleasure. The different vowel-sounds can thus be produced by employing the proper combinations.

The same apparatus served for establishing the principle (§ 925), that the character of a musical sound depends only on *constitution*, irrespective of change of phase.

928. Phonograph.—Mr. Edison of New York has been successful in constructing an instrument which can reproduce articulate sounds spoken into it. The voice of the speaker is directed into a funnel, which converges the sonorous waves upon a diaphragm carrying a style. The vibrations of the diaphragm are impressed by means of this style upon a sheet of tin-foil, which is fixed on the outside of a cylinder to which a spiral motion is given as in the vibroscope (Fig. 616). After this has been done, the cylinder with the tin-foil on it is shifted back to its original position, the style is brought into contact

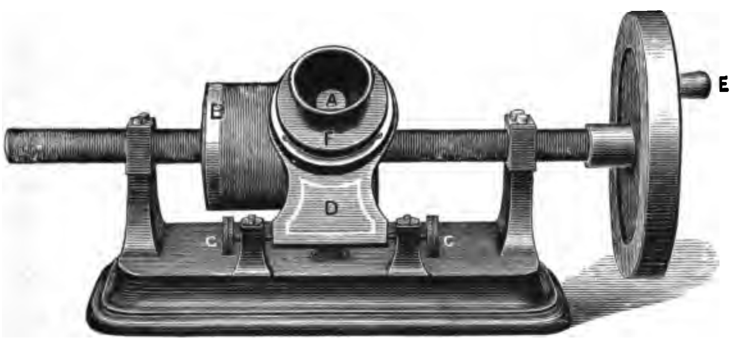


Fig. 640.—Phonograph.

with the tin-foil as at first, and the cylinder is then turned as before. The indented record is thus passed beneath the style, and forces it and the attached diaphragm to execute movements resembling their original movements. The diaphragm accordingly emits sounds which are imitations of those previously spoken to it. Tunes sung into the funnel are thus reproduced with great fidelity, and sentences clearly spoken into it are reproduced with sufficient distinctness to be understood.

The instrument is represented in Fig. 640. By turning the handle E, which is attached to a massive fly-wheel, the cylinder B is made

to revolve and at the same time to travel longitudinally, as the axle on which it is mounted is a screw working in a fixed nut. The surface of the cylinder is also fluted screw-fashion, the distance between its flutings being the same as the distance between the threads on the axle. A is the diaphragm, of thin sheet-iron, having the style fixed to its centre but not visible in the figure. The diaphragm and funnel are carried by the frame D, which turns on a hinge at the bottom. C C are adjusting screws for bringing the style exactly opposite the centre of the groove on the cylinder, and another screw is provided beneath the frame D, for making the style project so far as to indent the tin-foil without piercing it. The tin-foil is put round the cylinder, and lightly fastened with cement, so that it can be quickly taken off and changed.

In another form of the instrument, the rotation of the cylinder is effected by means of a driving weight and governor, which give it a constant velocity. This is a great advantage in reproducing music, but is of little or no benefit for speech.

CHAPTER LXVI.

CONSONANCE, DISSONANCE, AND RESULTANT TONES.

929. Concord and Discord.—Every one not utterly destitute of musical ear is familiar with the fact that certain notes, when sounded together, produce a pleasing effect by their combination, while certain others produce an unpleasing effect. The combination of two or more notes, when agreeable, is called *concord* or *consonance;* when disagreeable, *discord* or *dissonance.* The distinction is found to depend almost entirely on difference of pitch, that is, on relative frequency of vibration; so that the epithets consonant and dissonant can with propriety be applied to intervals.

The following intervals are consonant: unison (1 : 1), octave (1 : 2), octave + fifth (1 : 3), double octave (1 : 4), fifth (2 : 3), fourth (3 : 4).

The major third (4 : 5) and major sixth (3 : 5), together with the minor third (5 : 6) and minor sixth (5 : 8), are less perfect in their consonance.

The second and the seventh, whether major or minor, are dissonant intervals, whatever system of temperament be employed, as are also an indefinite number of other intervals not recognized in music.

Besides the difference as regards pleasing or unpleasing effect, it is to be remarked that consonant intervals can be identified by the ear with much greater accuracy than those which are dissonant. Musical instruments are generally tuned by octaves and fifths, because very slight errors of excess or defect in these intervals are easily detected by the ear. To tune a piano by the mere comparison of successive notes would be beyond the power of the most skilful musician. A sharply marked interval is always a consonant interval.

930. **Jarring Effect of Dissonance.**—According to the theory propounded by Helmholtz, the unpleasant effect of a dissonant interval consists essentially in the production of beats. These have a jarring effect upon the auditory apparatus, which becomes increasingly disagreeable as the beats increase in frequency up to a certain limit (about 33 per second for notes of medium pitch), and becomes gradually less disagreeable as the frequency is still further increased. The sensation produced by beats is comparable to that which the eye experiences from the *bobbing* of a gas flame in a room lighted by it; but the frequency which entails the maximum of annoyance is smaller for the eye than for the ear, on account of the greater persistence of visible impressions. The annoyance must evidently cease when the succession becomes so rapid as to produce the effect of a continuous impression.

We have already (§ 888) described a mode of producing beats with any degree of frequency at pleasure; and this experiment is one of the main foundations on which Helmholtz bases his view.

931. **Beats of Harmonics.**—The beats in the experiment above alluded to, are produced by the imperfect unison of two notes, and indicate the number of vibrations gained by one note upon the other. Their existence is easily and completely explained by the considerations adduced in § 888. But it is well known to musicians, and easily established by experiment, that beats are also produced between notes whose interval is approximately an octave, a fifth, or some other consonance; and that, in these cases also, the beats become more rapid as the interval becomes more faulty.

These beats are ascribed by Helmholtz to the common harmonic of the two fundamental notes. For example, in the case of the fifth (2 : 3), the third tone of the lower note would be identical with the second tone of the upper, if the interval were exact; and the beats which occur are due to the imperfect unison consequent on the deviation from exact truth. All beats are thus explained as due to imperfect *unison.*

This explanation is not merely conjectural, but is established by the following proofs:—

1. When an arrangement is employed by which the fifth is made false by a known amount, the number of beats is found to agree with the above explanation. Thus, if the interval is made to correspond to the ratio 200 : 301, it is observed that there are 2 beats to every 200 vibrations of the lower note. Now the harmonics which

are in approximate unison are represented by 600 and 602, and the difference of these is 2.

2. When the resonator corresponding to this common harmonic is held to the ear, it responds to the beats, showing that this harmonic is undergoing variations of strength; but when a resonator corresponding to either of the fundamental notes is employed, it does not respond to the beats, but indicates steady continuance of its appropriate note.

3. By a careful exercise of attention, a person with a good ear can hear, without any artificial aids, that it is the common harmonic which undergoes variations of intensity, and that the fundamental notes continue steady.

932. Beating Notes must be Near Together.—In order that two simple tones may yield audible beats, it is necessary that the musical interval between them should be small; in other words, that the ratio of their frequencies of vibration should be nearly equal to unity. Two simple notes of 300 and 320 vibrations per second will yield 20 beats in a second, and will be eminently discordant, the interval between them being only a semitone (15 : 16), but simple notes of 40 and 60 vibrations per second will not give beats, the interval between them being a fifth (2 : 3). The wider the interval between two simple notes, the feebler will be their beats; and accordingly, for a given frequency of beats, the harshness of the effect increases with the nearness of the notes to each other on the musical scale.[1] By taking joint account of the number of beats and the nearness of the beating tones, Helmholtz has endeavoured to express numerically the severity of the discords resulting from the combination of the note C of 256 vibrations per second with any possible note lying within an octave on the upper side of it, a particular constitution (approximately that of the violin) being assumed for both notes. He finds a complete absence of discord for the intervals of unison, the octave, and the fifth, and very small amounts of discord for the fourth, the sixth, and the third. By far the worst discords are found for the intervals of the semitone and major seventh,

[1] The explanation adopted by Helmholtz is, that a certain part of the ear—the *membrana basilaris*—is composed of tightly stretched elastic fibres, each of which is attuned to a particular simple tone, and is thrown into vibration when this tone, or one nearly in unison with it, is sounded. Two tones in approximate unison, when sounded together, affect several fibres in common, and cause them to beat. Tones not in approximate unison affect entirely distinct sets of fibres, and thus cannot produce interference.

and the next worst are for intervals a little greater or less than the fifth.

933. Imperfect Concord.—When there is a complete absence of discord between two notes, they are said to form a perfect concord. The intervals unison, fifth, octave, octave + fifth, and the interval from any note to any of its harmonics, are of this class. The third, fourth, and sixth are instances of imperfect concord. Suppose, for example, that the two notes sounded together are C of 256 and E of 320 vibrations per second, the interval between these notes being a true major third (4 : 5); and suppose each of these notes to consist of the first six simple tones.

The first six multiples of 4 are

4, 8, 12, 16, 20, 24.

The first six multiples of 5 are

5, 10, 15, 20, 25, 30.

In searching for elements of discord, we select (one from each line) two multiples differing by unity.

Those which satisfy this condition are

4 and 5; 16 and 15; 24 and 25.

But the first pair (4 and 5) may be neglected, because their ratio differs too much from unity. Discordance will result from each of the two remaining pairs; that is to say, the 4th element of the lower of our two given notes is in discordance with the 3d element of the upper; and the 6th element of the lower is in discordance with the 5th element of the higher. To find the frequencies of the beats, we must multiply all these numbers by 64, since 256 is 4 times 64, and 320 is 5 times 64. Instead of a difference of 1, we shall then find a difference of 64, that is to say, the number of beats per second is 64 in the case of each of the two discordant combinations which we have been considering.

934. Resultant Tones.—Under certain conditions it is found that two notes, when sounded together, produce by their combination other notes, which are not constituents of either. They are called *resultant tones*, and are of two kinds, *difference-tones* and *summation-tones*. A difference-tone has a frequency of vibration which is the difference of the frequencies of its components. A summation-tone has a frequency of vibration which is the sum of the

frequencies of its components. As the components may either be fundamental tones or overtones, two notes which are rich in harmonies may yield, by their combination, a large number of resultant tones.

The difference-tones were observed in the last century by Sorge and by Tartini, and were, until recently, attributed to beats. The frequency of beats is always the difference of the frequencies of vibration of the two elementary tones which produce them; and it was supposed that a rapid succession of beats produced a note of pitch corresponding to this frequency.

This explanation, if admitted, would furnish an exception to what otherwise appears to be the universal law, that every *elementary tone* arises from a corresponding *simple vibration.*[1] Such an exception should not be admitted without necessity; and in the present instance it is not only unnecessary, but also insufficient, inasmuch as it fails to render any account of the summation-tones.

Helmholtz has shown, by a mathematical investigation, that when two systems of simple waves agitate the same mass of air, their mutual influence must, according to the recognized laws of dynamics, give rise to two derived systems, having frequencies which are respectively the sum and the difference of the frequencies of the two primary systems. Both classes of resultant tones are thus completely accounted for.

The resultant tones—especially the summation-tones, which are fainter than the others—are only audible when the primary tones are loud; for their existence depends upon small quantities of the second order, the amplitudes of the primaries being regarded (in comparison with the wave-lengths) as small quantities of the first order.

If any further proof be required that the difference tones are not due to the coalescence of beats, it is furnished by the fact that, under favourable conditions, the rattle of the beats and the booming of the difference-tones can both be heard together.

935. Beats due to Resultant Tones.—The existence of resultant tones serves to explain, in certain cases, the production of beats between notes which are wanting in harmonics. For example, if two *simple* sounds, of 100 and 201 vibrations per second respectively, are sounded together, one beat per second will be produced between

[1] The discovery of this law is due to Ohm.

the difference-tone of 101 vibrations and the primary tone of 100 vibrations. By the beats to which they thus give rise, resultant tones exercise an influence on consonance and dissonance.

Resultant tones, when sufficiently loud, are themselves capable of performing the part of primaries, and yielding what are called *resultant tones of the second order*, by their combination with other primaries. Several higher orders of resultant tones can, under peculiarly favourable circumstances, be sometimes detected.

OPTICS.

CHAPTER LXVII.

PROPAGATION OF LIGHT.

936. Light.—Light is the immediate external cause of our visual impressions. Objects, except such as are styled *self-luminous*, become invisible when brought into a dark room. The presence of something additional is necessary to render them visible, and that mysterious agent, whatever its real nature may be, we call *light*.

Light, like sound, is believed to consist in vibration; but it does not, like sound, require the presence of air or other gross matter to enable its vibrations to be propagated from the source to the percipient. When we exhaust a receiver, objects in its interior do not become less visible; and the light of the heavenly bodies is not prevented from reaching us by the highly vacuous spaces which lie between.

It seems necessary to assume the existence of a medium far more subtle than ordinary matter; a medium which pervades alike the most vacuous spaces and the interior of all bodies, whether solid, liquid, or gaseous; and which is so highly elastic, in proportion to its density, that it is capable of transmitting vibrations with a velocity enormously transcending that of sound.

This hypothetical medium is called *æther*. From the extreme facility with which bodies move about in it, we might be disposed to call it a subtle *fluid;* but the undulations which it serves to propagate are not such as can be propagated by fluids. Its elastic properties are rather those of a solid; and its waves are analogous to the pulses which travel along the wires of a piano rather than to the waves of extension and compression by which sound is propagated through air. *Luminous vibrations are transverse, while those of sound are longitudinal.*

A self-luminous body, such as a red-hot poker or the flame of a

candle, is in a peculiar state of vibration. This vibration is communicated to the surrounding æther, and is thus propagated to the eye, enabling us to see the body. In the majority of cases, however, we see bodies not by their own but by reflected light; and we are enabled to recognize the various kinds of bodies by the different modifications which light undergoes in reflection from their surfaces.

As all bodies can become sonorous, so also all bodies can become self-luminous. To render them so, it is only necessary to raise them to a sufficiently high temperature, whether by the communication of heat from a furnace, or by the passage of an electric current, or by causing them to enter into chemical combination. It is to chemical combination, in the active form of combustion, that we are indebted for all the sources of artificial light in ordinary use.

The vibrations of the æther are capable of producing other effects besides illumination. They constitute what is called radiant heat, and they are also capable of producing chemical effects, as in photography. Vibrations of high frequency, or short period, are the most active chemically. Those of low frequency or long period have usually the most powerful heating effects; while those which affect the eye with the sense of light are of moderate frequency.

937. Rectilinear Propagation of Light.—All the remarks which have been made respecting the relations between period, frequency, and wave-length, in the case of sound, are equally applicable to light, inasmuch as all kinds of luminous waves (like all kinds of sonorous waves) have sensibly the same velocity in air; but this velocity is many hundreds of thousands of times greater for light than for sound, and the wave-lengths of light are at the same time very much shorter than those of sound. Frequency, being the quotient of velocity by wave-length, is accordingly about a million of millions of times greater for light than for sound. The colour of lowest pitch is deep red, its frequency being about 400 million million vibrations per second, and its wave-length in air 760 millionths of a millimetre. The colour of highest pitch is deep violet; its frequency is about 760 million million vibrations per second, and its wave-length in air 400 millionths of a millimetre. It thus appears that the range of seeing is much smaller than that of hearing, being only about one octave.

The excessive shortness of luminous as compared with sonorous waves is closely connected with the strength of the shadows cast by a light, as compared with the very moderate loss of intensity produced by interposing an obstacle in the case of sound. Sound may,

for ordinary purposes, be said to be capable of turning a corner, and light to be only capable of travelling in straight lines. The latter fact may be established by such an arrangement as is represented in Fig. 641. Two screens, each pierced with a hole, are arranged so that these holes are in a line with the flame of a candle. An eye placed in this line, behind the screens, is then able to see the flame; but a slight lateral displacement, either of the eye, the candle, or either of the screens, puts the flame out of sight. It is to be noted that, in this experiment, the same medium (air) extends from the eye to the candle. We shall hereafter find that, when light has to pass from one medium to another, it is often bent out of a straight line.

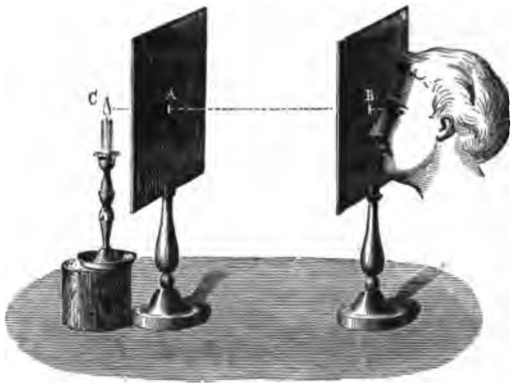


Fig. 641.—Rectilinear Propagation.

We have said that the strength of light-shadows as compared with sound-shadows is connected with the shortness of luminous waves. Theory shows that, if light is transmitted through a hole or slit, whose diameter is a very large multiple of the length of a light-wave, a strong shadow should be cast in all oblique directions; but that, if the hole or slit is so narrow that its diameter is comparable to the length of a wave, a large area not in the direct path of the beam will be illuminated. The experiment is easily performed in a dark room, by admitting sunlight through an exceedingly fine slit, and receiving it on a screen of white paper. The illuminated area will be marked with coloured bands, called diffraction-fringes; and if the slit is made narrower, these bands become wider.

On the other hand, Colladon, in his experiments on the transmission of sound through the water of the Lake of Geneva, established the presence of a very sharply defined sound-shadow in the water, behind the end of a projecting wall.

For the present we shall ignore diffraction,[1] and confine our atten-

[1] See Chap. lxxiv.

tion to the numerous phenomena which result from the rectilinear propagation of light.

938. Images produced by Small Apertures.—If a white screen is placed opposite a hole in the shutter of a room otherwise quite dark,

Fig. 642.—Image formed by Small Aperture.

an inverted picture of the external landscape will be formed upon it, in the natural colours. The outlines will be sharper in proportion as the hole is smaller, and distant objects will be more distinctly represented than those which are very near.

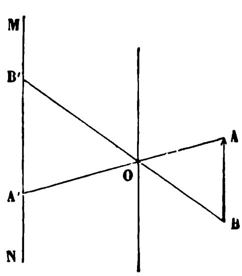


Fig. 643.—Explanation.

These results are easily explained. Consider, in fact, an external object A B (Fig. 643), and let O be the hole in the shutter. The point A sends rays in all directions into space, and among them a small pencil, which, after passing through the opening O, falls upon the screen at A′. A′ receives light from no other point but A, and A sends light to no part of the screen except A′. The colour and brightness of the spot A′ will accordingly depend upon the colour and brightness of A; in other words, A′ will be the

image of A. In like manner B′ will be the image of B, and points of the object between A and B will have their images between A′ and B′. An inverted image A′B′ will thus be formed of the object AB.

As the image thus formed of an external point is not a point, but a spot, whose size increases with that of the opening, there must always be a little blurring of the outlines from the overlapping of the spots which represent neighbouring points; but this will be comparatively slight if the opening is very small.

An experiment, substantially the same as the above, may be performed by piercing a card with a large pin-hole, and holding it between

Fig. 644.—Image formed by Hole in a Card.

a candle and a screen, as in Fig. 644. An inverted image of the candle will thus be formed upon the screen.

When the sun shines through a small hole into a room with the blinds down (Fig. 645), the cone of rays thus admitted is easily traced by the lighting up of the particles of dust which lie in its course. The image of the sun which is formed at its further extremity will be either circular or elliptical, according as the incidence of the rays is normal or oblique. Fine images of the sun are sometimes thus formed by the chinks of a venetian-blind, especially when the sun is low, and there is a white wall opposite to receive the

image. In these circumstances it is sometimes possible to detect the presence of spots on the sun by examining the image.

When the sun's rays shine through the foliage of a tree (Fig. 646), the spots of light which they form upon the ground are always round or oval, whatever may be the shape of the interstices through which they have passed, provided always that these interstices are small. When the sun is undergoing eclipse, the progress of the eclipse can

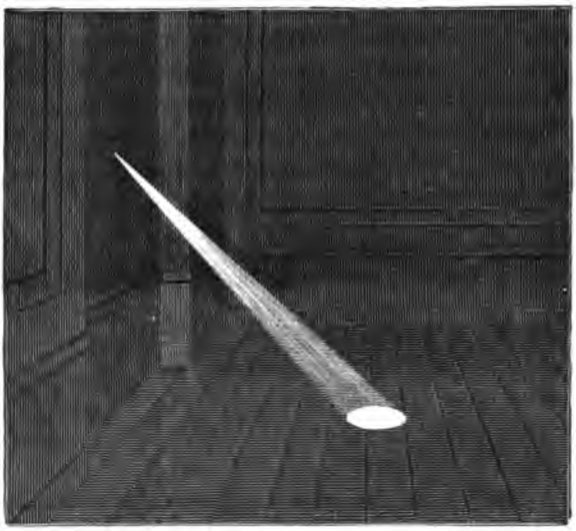


Fig. 645.—Conical Sunbeam.

be traced by watching the shape of these images, which resembles that of the uneclipsed portion of the sun's disc.

939. Theory of Shadows.—The rectilinear propagation of light is the foundation of the geometry of shadows. Let the source of light be a luminous point, and let an opaque body be placed so as to intercept a portion of its rays (Fig. 647). If we construct a conical surface touching the body all round, and having its vertex at the luminous point, it is evident that all the space within this surface on the further side of the opaque body is completely screened from the rays. The cone thus constructed is called the shadow-cone, and its intersection with any surface behind the opaque body defines the shadow cast upon that surface. In the case which we have been supposing—that of a luminous point—the shadow-cone and the shadow itself will be sharply defined.

Fig. 646.—Images of Sun formed by Foliage.

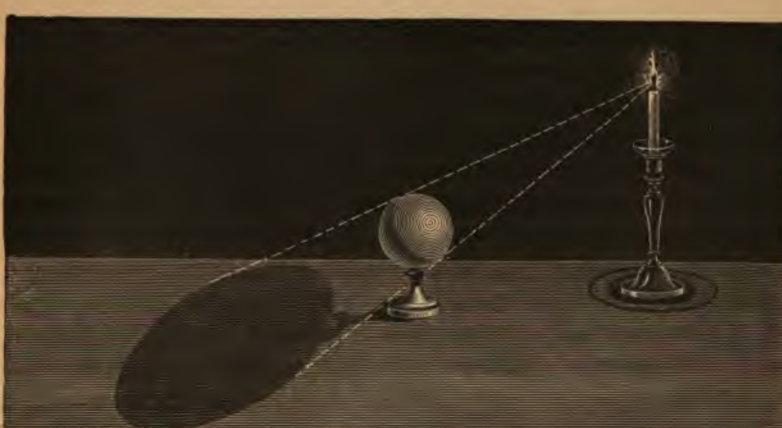


Fig. 647.—Shadow.

Actual sources of light, however, are not mere luminous points, but have finite dimensions. Hence some complication arises. Consider, in fact (Fig. 648), a luminous body situated between two opaque bodies, one of them larger, and the other smaller than itself. Conceive a cone touching the luminous body and either of the opaque bodies *externally*. This will be the cone of *total shadow*, or the cone of the *umbra*. All points lying within it are completely excluded from view of the luminous body. This cone narrows or enlarges as it recedes, according as the opaque body is smaller or larger than the luminous body. In the former case it terminates at a finite distance. In the latter case it extends to infinite distance.

Now conceive a double cone touching the luminous body and either of the opaque bodies *internally*. This cone will be wider than the cone of total shadow, and will include it. It is called the

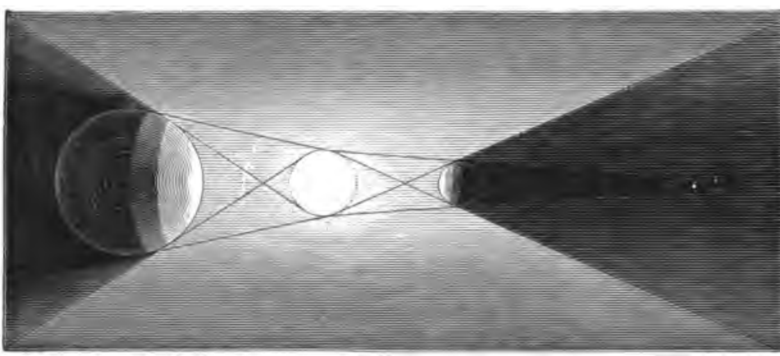


Fig. 648.—Umbra and Penumbra.

cone of *partial shadow*, or the cone of the *penumbra*. All points lying within it are excluded from the view of some portion of the luminous body, and are thus partially shaded by the opaque body. If they are near its outer boundary, they are very slightly shaded. If they are so far within it as to be near the total shadow, they are almost completely shaded. Accordingly, if the shadow of the opaque body is received upon a screen, it will not have sharply defined edges, but will show a gradual transition from the total shadow which covers a finite central area to a complete absence of shadow at the outer boundary of the penumbra. Thus neither the edges of the umbra nor those of the penumbra are sharply defined.

The umbra and penumbra show themselves on the surface of the

opaque body itself, the line of contact of the umbral cone being further back from the source of light than the line of contact of the penumbral cone. The zone between these two lines is in partial shadow, and separates the portion of the surface which is in total shadow from the part which is not shaded at all.

940. Velocity of Light.—Luminous undulations, unlike those of sound, advance with a velocity which may fairly be styled inconceivable, being about 300 million metres per second, or 186,000 miles per second. As the circumference of the earth is only 40 million metres, light would travel seven and a half times round the earth in a second.

Hopeless as it might appear to attempt the measurement of such an enormous velocity by mere terrestrial experiments, the feat has actually been performed, and that by two distinct methods. In Fizeau's experiments the distance between the two experimental stations was about 5½ miles. In Foucault's experiments the whole apparatus was contained in one room, and the movement of light within this room served to determine the velocity.

We will first describe Fizeau's experiment.

941. Fizeau's Experiment.—Imagine a source of light placed directly in front of a plane mirror, at a great distance. The mirror will send back a reflected beam along the line of the incident beam, and an observer stationed behind the source will see its image in the mirror as a luminous point.

Now imagine a toothed-wheel, with its plane perpendicular to the path of the beam, revolving uniformly in front of the source, in such a position that its teeth pass directly between the source of light and the mirror. The incident beam will be stopped by the teeth, as they successively come up, but will pass through the spaces between them. Now the velocity of the wheel may be such that the light which has thus passed through a space shall be reflected back from the mirror just in time to meet a tooth and be stopped. In this case it will not reach the observer's eye, and the image may thus become permanently invisible to him. From the velocity of the wheel, and the number of its teeth, it will be possible to compute the time occupied by the light in travelling from the wheel to the mirror, and back again. If the velocity of the wheel is such that the light is sometimes intercepted on its return, and sometimes allowed to pass, the image will appear steadily visible, in consequence of the persistence of impressions on the retina, but with a

loss of brightness proportioned to the time that the light is intercepted. The wheel employed by Fizeau had 720 teeth, the distance between the two stations was 8663 metres, and 12·6 revolutions per second produced disappearance of the image. The width of the teeth being equal to the width of the spaces, the time required to turn through the width of a tooth was $\frac{1}{2} \times \frac{1}{720} \times \frac{1}{12 \cdot 6}$ of a second, that is $\frac{1}{18144}$ of a second.

In this time the light travelled a distance of $2 \times 8663 = 17326$ metres. The distance traversed by light in a second would therefore be $17{,}326 \times 18{,}144 = 314{,}262{,}944$ metres. This determination of M. Fizeau's is believed to be somewhat in excess of the truth.

A double velocity of the wheel would allow the reflected beam to pass through the space succeeding that through which the incident beam had passed; a triple velocity would again produce total eclipse, and so on. Several independent determinations of the velocity of light may thus be obtained.

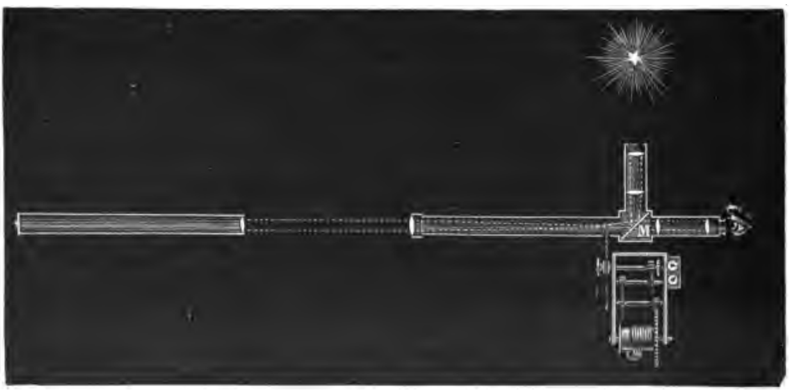


Fig. 649.—Fizeau's Experiment.

Thus far, we have merely indicated the principle of calculation. It will easily be understood that special means were necessary to prevent scattering of the light, and render the image visible at so great a distance. Fig. 649 will serve to give an idea of the apparatus actually employed.

A beam of light from a lamp, after passing through a lens, falls on a plate of unsilvered glass M, placed at an angle of 45°, by which it is reflected along the tube of a telescope; the object-glass of the telescope is so adjusted as to render the rays parallel on emergence, and in this condition they traverse the interval

between the two stations. At the second station they are collected by a lens, which brings them to a focus on the surface of a mirror, which sends them back along the same course by which they came. A portion of the light thus sent back to the glass plate M passes through it, and is viewed by the observer through an eye-piece.

The wheel R is driven by clock-work. Figs. 650, 651, 652 respec-

Fig. 650.—Wheel at Rest.

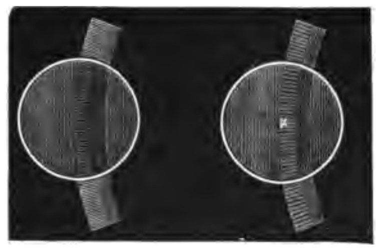


Fig. 651.—Total Eclipse. **Fig. 652.—Partial Eclipse.**

tively represent the appearance of the luminous point as seen between the teeth of the wheel when not revolving, the total eclipse produced by an appropriate speed of rotation, and the partial eclipse produced by a different speed.

More recently M. Cornu has carried out an extensive series of experiments on the same plan, with more powerful appliances, the distance between the two stations being 23 kilometres, and the extinctions being carried to the 21st order. His result is that the velocity of light (in millions of metres per second) is 300·33 in air, or 300·4 *in vacuo*.

942. Foucault's Experiment.—Foucault employed the principle of the rotating mirror, first adopted by Wheatstone in his experiments on the duration of the electric spark and the velocity of electricity (§ 591, 636). The following was the construction of his original apparatus:—

A beam of light enters a room by a square hole, which has a fine platinum wire stretched across it, to serve as a mark; it is then concentrated by an achromatic lens, and, before coming to a focus, falls upon a plane mirror, revolving about an axis in its own plane. In one part of the revolution the reflected beam is directed upon a concave mirror, whose centre of curvature is in the axis of rotation, so that the beam is reflected back to the revolving mirror, and

thence back to the hole at which it first entered. Before reaching the hole, it has to traverse a sheet of glass, placed at an angle of 45°, which reflects a portion of it towards the observer's eye; and the image which it forms (an image of the platinum wire) is viewed through a powerful eye-piece. The image is only formed during a small part of each revolution; but when 30 turns are made per second, the appearance presented, in consequence of the persistence of impressions, is that of a permanent image occupying a fixed position. When the speed is considerably greater, the mirror turns through a sensible angle while the light is travelling from it to the concave mirror and back again, and a sensible displacement of the image is accordingly observed. The actual speed of rotation was from 700 to 800 revolutions per second.[1]

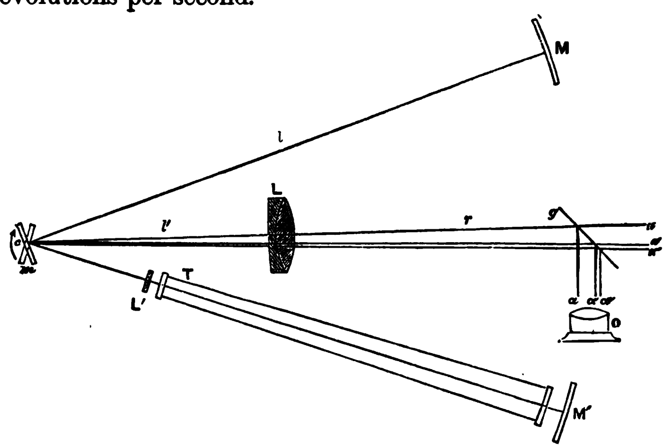


Fig. 653.—Foucault's Experiment.

On interposing a tube filled with water between the two mirrors, it was found that the displacement was increased, showing that a longer time was occupied in traversing the water than in traversing the same length of air.

This result, as we shall have occasion to point out later, is very important as confirming the undulatory theory and disproving the emission theory of light.

In Fig. 653, *a* is the position of the platinum wire, L is the achromatic lens, *m* the revolving mirror, *c* the axis of revolution, M

[1] It was found that, at this high speed, the amalgam at the back of ordinary looking-glasses was driven off by centrifugal force. The mirror actually employed was silvered in front with real silver.

the concave mirror, a' the image of the platinum wire, displaced from a in virtue of the rotation of the mirror; α, α' images of a, a', formed by the glass plate g, and viewed through the eye-piece O.

M′ is a second concave mirror, at the same distance as M from the revolving mirror; T is a tube filled with water, and having plane glass ends, and L′ a lens necessary for completing the focal adjustment; a'' and α'' are the images formed by the light which has traversed the water.[1]

Foucault's experiment, as thus described, was performed in 1850 very shortly after that of Fizeau, and was mainly designed for giving a relative determination of the velocities in air and water. Foucault subsequently adapted it to absolute measurement, and determined the velocity in air to be 298 million metres per second.

943. Later Determinations.—Captain Michelson of the United States navy carried out in 1879 and 1882 two excellent series of experiments, in which the lens L was placed not between the slit and the revolving mirror but between the revolving and the fixed mirror, in such a position that the sum of the distances of the slit and lens from the revolving mirror was a very little greater than the focal length of the lens. The image of the slit was accordingly formed at a very great distance on the other side of the lens, and it was at this distance that the fixed mirror M was placed. The focal length of the lens was 150 ft., and the distance between the two mirrors nearly 2000 ft. The measured deviation of the image of

[1] The distances are such that La and Lc+cM are conjugate focal distances with respect to the lens L. An image of the wire a is thus formed at M, and an image of this image is formed at a, the mirror being supposed stationary; and this relation holds not only for the central point of the concave mirror, but for any part of it on which the light may happen to fall at the instant considered.

Let l denote the distance cM between the revolving and the fixed mirror, l' the distance cL of the revolving mirror from the centre of the lens, r the distance αL of the platinum wire from the centre of the lens, n the number of revolutions per second, V the space traversed by light in a second, t the time occupied by light in travelling from one mirror to the other and back, θ the angle turned by the mirror in this time, and δ the angle subtended at the centre of the lens by the distance $a\,a'$ between the wire and its displaced image.

Then obviously $t=\frac{2\,l}{V}$, but also $t=\frac{\theta}{2\,\pi\,n}$; hence $V=\frac{4\,\pi\,n\,l}{\theta}$.

Now the distance between the two images (corresponding to a, a' respectively) at the back of the revolving mirror is $(l+l')\,\delta$, and is also $2\,\theta\,l$ (§ 964). Hence $\theta=\frac{(l+l')\,\delta}{2\,l}$, and $V=\frac{8\,\pi\,n\,l^2}{(l+l')\delta}$. The observed distance $\alpha\,\alpha'$ between the two images is equal to the distance between a, a', that is to $r\,\delta$. Calling this distance d, we have finally,

$$V=\frac{8\,\pi\,n\,l^2\,r}{l(l+l')\,d}.$$

the slit from the slit itself was due to the angle through which the mirror turned while light travelled over twice this distance, or nearly 4000 ft.; and the distance of the slit from the mirror being about 30 ft., the deviation of the image from the slit amounted to more than 133 millimetres, whereas the deviation obtained by Foucault was less than 1 millimetre. The velocity *in vacuo* finally deduced by Michelson was 299·91 from the first series, and 299·85 from the second series.

A still better determination was made in 1882 by Professor Newcomb of the United States Naval Observatory, Washington, the distance between the revolving and the fixed mirror being in some of the observations 2550 metres, and in others 3720 metres. The method employed was in principle the same as in Foucault's original experiment. The revolving mirror was four-sided, like that in Fig. 639, p. 937, but made of polished steel, and driven by a blast of air impinging on vanes. The speed of revolution was measured by a self-recording apparatus which, by breaking an electric current, made a mark once in every 28 revolutions upon paper, on which a mark was also made every second by a chronograph.

The source of light was a slit illuminated by the rays of the sun reflected from a heliostat. The slit was in the principal focus of a collimating lens, so that a parallel beam of light fell upon the revolving mirror, and was reflected to and from the distant station. On its return it was reflected by the revolving mirror into an observing telescope furnished with two parallel wires near together in the focus of its eye-piece, between which the image was made to fall. This telescope was so mounted that, while always directed centrally on the revolving mirror, it could be moved through about 4° on each side of the position of no deviation; and in actual use it was moved into such a position that the displaced image of the slit could be kept steadily between the two parallel wires, the regulation of the displacement being effected by means of two cords which governed the blast of air. The deviation amounted in some of the experiments to 3° on each side, the arrangements being such that the mirror could be turned either way. The result deduced was a velocity *in vacuo* of 299·86 million metres per second (about 186,300 miles per second), which may be accepted as the best determination yet made.

944. Velocity of Light deduced from Observations of the Eclipses of Jupiter's Satellites.—The fact that light occupies a sensible time in travelling over celestial distances, was first established about 1675,

by Roemer, a Danish astronomer, who also made the first computation of its velocity. He was led to this discovery by comparing the observed times of the eclipses of Jupiter's first satellite, as contained in records extending over many successive years.

The four satellites of Jupiter revolve nearly in the plane of the planet's orbit, and undergo very frequent eclipse by entering the cone of total shadow cast by Jupiter. The satellites and their eclipses are easily seen, even with telescopes of very moderate power; and being visible at the same absolute time at all parts of the earth's surface at which they are visible at all, they serve as signals for comparing local time at different places, and thus for determining longitudes. The first satellite (that is, the one nearest to Jupiter), from its more rapid motion and shorter time of revolution, affords both the best and the most frequent signals. The interval of time between two successive eclipses of this satellite is about 42½ hours, but was found by Roemer to vary by a regular law according to the position of the earth with respect to Jupiter. It is longest when the earth is increasing its distance from Jupiter most rapidly, and is shortest when the earth is diminishing its distance most rapidly. Starting from the time when the earth is nearest to Jupiter, as at T, J (Fig. 654), the intervals between successive eclipses are always longer than the mean value, until the greatest distance has been attained, as at T′ J′, and the sum of the excesses amounts to 16 min. 26·6 sec. From this time until the nearest distance is again attained, as at T″, J″, the intervals are always shorter than the mean, and the sum of the defects amounts to 16 min. 26·6 sec. It is evident, then, that the eclipses are visible 16 m. 26·6 s. earlier at the nearest than at the remotest point of the earth's orbit; in other words, that this is the time required for the propagation of light across the diameter of the orbit. Taking this diameter as 184 millions of miles,[1] we have a resulting velocity of about 186,500 miles per second.

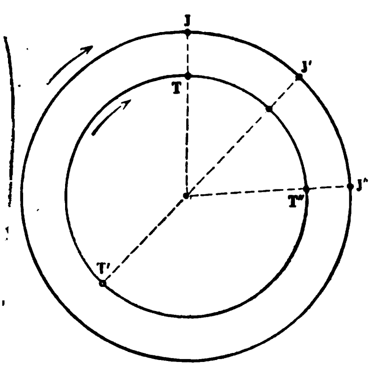


Fig. 654.—Earth and Jupiter.

945. Velocity of Light deduced from Aberration.—About fifty

[1] The sun's mean distance from the earth was, until recently, estimated at 95 millions of miles. It is now estimated at 92 or 92½ millions.

years after Roemer's discovery, Bradley, the English astronomer, employed the velocity of light to explain the astronomical phenomenon called *aberration*. This consists in a regular periodic displacement of the stars as seen from the earth, the period of the displacement being a year. If the direction in which the earth is moving in its orbit at any instant be regarded as the *forward* direction every star constantly appears on the forward side of its true place, so that, as the earth moves once round its orbit in a year, each star describes in this time a small apparent orbit about its true place.

The phenomenon is explained in the same way as the familiar fact, that a shower of rain falling vertically, seems, to a person running forwards, to be coming in his face. The relative motion of the rain-drops with respect to his body, is found by compounding the actual velocity of the drops (whether vertical or oblique) with a velocity equal and opposite to that with which he runs. Thus if A B (Fig. 655) represents the velocity with which he runs, and C A, the true velocity of the drops, the apparent velocity of the drops will be represented by D A. If a tube pointed along A D moves forward parallel to itself with the velocity A B, a drop entering at its upper end will pass through its whole length without wetting its sides; for while the drop is falling along D B (we suppose with uniform velocity) the tube moves along A B, so that the lower end of the tube reaches B at the same time as the rain-drop.

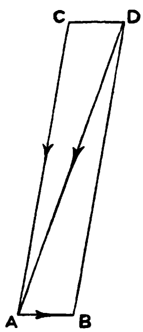


Fig. 655. Aberration.

In like manner, if A B is the velocity of the earth, and C A the velocity of light, a telescope must be pointed along A D to see a star which really lies in the direction of A C or B D produced. When the angle B A C is a right angle (in other words, when the star lies in a direction perpendicular to that in which the earth is moving), the angle C A D, which is called the aberration of the star, is $20''\cdot5$, and the tangent of this angle is the ratio of the velocity of the earth to the velocity of light. Hence it is found by computation that the velocity of light is about ten thousand times greater than that with which the earth moves in its orbit. The latter is easily computed, if the sun's distance is known, and is about $18\frac{1}{2}$ miles per second. Hence the velocity of light is about 185,000 miles per second. It will be noted that both these astronomical methods of computing the velocity of light, depend upon the knowledge of the sun's distance

from the earth, and that, if this distance is overestimated, the computed velocity of light will be too great in the same ratio.

Conversely, the velocity of light, as determined by Foucault's method, can be employed in connection either with aberration or the eclipses of the satellites, for computing the sun's distance; and the first correct determination of the sun's distance was, in fact, that deduced by Foucault from his own results.

946. Photometry.—Photometry is the measurement of the relative amounts of light emitted by different sources. The methods employed for this purpose all consist in determinations of the relative distances at which two sources produce equal intensities of illumination. The eye would be quite incompetent to measure the ratio of two unequal illuminations; but a pretty accurate judgment can be formed as to equality or inequality of illumination, at least when the surfaces compared are similar, and the lights by which they are illuminated are of the same colour. The law of inverse squares is always made the basis of the resulting calculations; and this law may itself be verified by showing that the illumination produced by one candle at a given distance is equal to that produced by four candles at a distance twice as great.

947. Bouguer's Photometer.—Bouguer's photometer consists of a

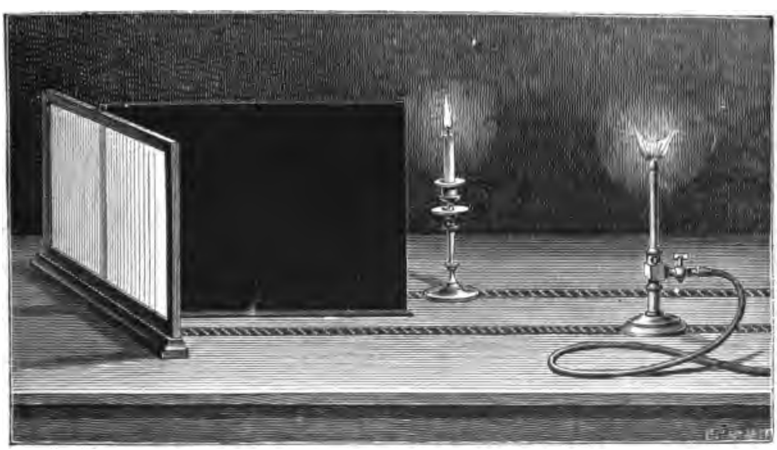


Fig. 656.—Bouguer's Photometer.

semi-transparent screen, of white tissue paper, ground glass, or thin white porcelain, divided into two parts by an opaque partition at right angles to it. The two lamps which are to be compared are

placed one on each side of this partition, so that each of them illuminates one-half of the transparent screen. The distances of the two lamps are adjusted until the two portions of the screen, as seen from the back, appear equally bright. The distances are then measured, and their squares are assumed to be directly proportional to the illuminating powers of the lamps.

948. Rumford's Photometer.—Rumford's photometer is based on the comparison of shadows. A cylindric rod is so placed that each of the two lamps casts a shadow of it on a screen; and the distances

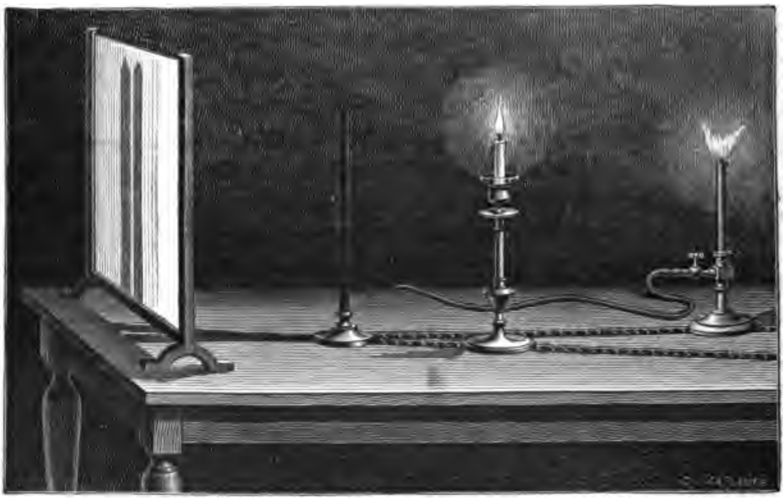


Fig. 657.—Rumford's Photometer.

are adjusted until the two shadows are equally dark. As the shadow thrown by one lamp is illuminated by the other lamp, the comparison of shadows is really a comparison of illuminations.

949. Foucault's Photometer.—The two photometers just described are alike in principle. In each of them the two surfaces compared are illuminated each by one only of the sources of light. In Rumford's the remainder of the screen is illuminated by both. In Bouguer's it consists merely of an intervening strip which is illuminated by neither. If the partition is movable, the effect of moving it further from the screen will be to make this dark strip narrower until it disappears altogether; and if it be advanced still further, the two illuminated portions will overlap. In Foucault's photometer there is an adjusting screw, for the purpose of advancing the parti-

tion so far that the dark strip shall just vanish. The two illuminated portions, being then exactly contiguous, can be more easily and certainly compared.

950. Bunsen's Photometer.—In the instruments above described the two sources to be compared are both on the same side of the screen, and illuminate different portions of it. Bunsen introduced the plan of placing the sources on opposite sides of the screen, and making the screen consist of two parts, one of them more translucent than the other. In his original pattern the screen was a sheet of white paper, with a large grease spot in the centre. In Dr. Letheby's pattern it is composed of three sheets of paper, laid face to face, the middle one being very thin, and the other two being cut away in the centre, so that the central part of the screen consists of one thickness, and the outer part of three.

When such a screen is more strongly illuminated on one side than on the other, the more translucent part appears brighter than the less translucent when seen from the darker side, while the reverse appearance is presented on the brighter side. It is therefore the business of the observer so to adjust the distances that the central and circumferential parts appear equally bright. When they appear equally bright from one side they will also appear equally bright from the other; but as there is always some little difference of tint, the observer's judgment is aided by seeing both sides at once. This is accomplished in Dr. Letheby's photometer by means of two mirrors, one for each eye, as represented in the accompanying ground-plan (Fig. 658).

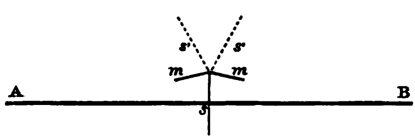


Fig. 658.—Letheby's Photometer.

s is the screen, and *n m* are the two mirrors, in which images *s′ s′* are seen by an observer in front. The frame which carries the screen and mirrors travels along a graduated bar A B, on which the distances of the screen from the two lights are indicated.

In all delicate photometric observations, the eye should be shielded from direct view of the lights, and, as much as possible, from all extraneous lights. The objects to be compared should be brighter than anything else in the field of view.

951. Photometers for very Powerful Lights.—In comparing two very unequal lights, for example, a powerful electric light and a

standard candle, it is scarcely possible to obtain an observing-room long enough for a direct comparison by any of the above methods. To overcome this difficulty a lens (either convex or concave), of short focal length, may be employed to form an image of the more powerful source near its principal focus. Then all the light which this source sends to the lens may be regarded as diverging from the image and filling a solid angle equal to that which the lens subtends at the image. In other words, the illuminations of the lens itself due to the source and the image are equal. Hence, if S and I are the distances of the source and image from the lens, the image is weaker than the source in the ratio of I^2 to S^2, and a direct comparison can be made between the light from the image and that from a standard candle. Thus, if a screen at distance D from the image has the same illumination from the image as from a candle at distance C on the other side, the image is equal to $\frac{D^2}{C^2}$ candles, and the source itself to $\frac{D^2}{C^2}\frac{S^2}{I^2}$ candles. A correction must, however, be applied to this result for the light lost by reflection at the surfaces of the lens.

CHAPTER LXVIII.

REFLECTION OF LIGHT.

952. Reflection.—If a beam of the sun's rays A B (Fig. 659) be aamitted through a small hole in the shutter of a dark room, and allowed to fall on a polished plane surface, it will be seen to continue its course in a different direction B C. This is an example of reflec-

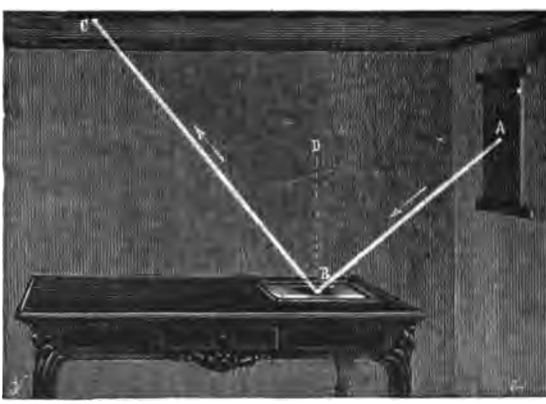


Fig. 659.—Reflection of Light.

tion. A B is called the incident beam, and B C the reflected beam. The angle A B D contained between an incident ray and the normal is called the angle of incidence; and the angle C B D contained between the corresponding reflected ray and the normal is called the angle of reflection. The plane A B D containing the incident ray and the normal is called the plane of incidence.

953. Laws of Reflection.—The reflection of light from polished surfaces takes place according to the following laws:—

1. The reflected ray lies in the plane of incidence.

2. The angle of reflection is equal to the angle of incidence.

These laws may be verified by means of the apparatus represented in Fig. 660. A vertical divided circle has a small polished plate fixed at its centre, at right angles to its plane, and two tubes travelling on its circumference with their axes always directed towards the centre. The zero of the divisions is the highest point of the circle, the plate being horizontal.

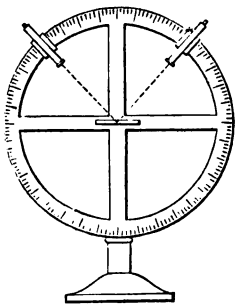


Fig. 660.—Verification of Laws of Reflection.

A source of light, such as the flame of a candle, is placed so that its rays shine through one of the tubes upon the plate at the centre. As the tubes are blackened internally, no light passes through except in a direction almost precisely parallel to the axis of the tube. The observer then looks through the other tube, and moves it along the circumference till he finds the position in which the reflected light is visible through it. On examining the graduations, it will be found that the two tubes are at the same distance from the zero point, on opposite sides. Hence the angles of incidence and reflection are equal. Moreover the plane of the circle is the plane of incidence, and this also contains the reflected rays. Both the laws are thus verified.

954. Artificial Horizon.—These laws furnish the basis of a method of observation which is frequently employed for determining the altitude of a star, and which, by the consistency of its results, furnishes a very rigorous proof of the laws.

A vertical divided circle (Fig. 661) is set in a vertical plane by proper adjustments. A telescope movable about the axis of the circle is pointed to a particular star, so that its line of collimation I′ S′ passes through the apparent place of the star. Another telescope,[1] similarly mounted on the other side of the circle, is directed downwards along the line I′R towards the image of the star as seen in a trough of mercury I. Assuming the truth of the laws of reflection as above stated, the altitude of the star is half the angle between the directions of the two telescopes; for the ray S I from the star to the mercury is parallel to the line S′ I′, by reason of the excessively great distance of the star; and since the rays S I, I R are equally inclined to the normal I N, which is a vertical line, the lines I′S, I′R are also equally inclined to the vertical, or, what is the same thing,

[1] In practice, a single telescope usually serves for both observations.

are equally inclined to a horizontal plane. A reflecting surface of mercury thus used is called a mercury horizon, or an *artificial*

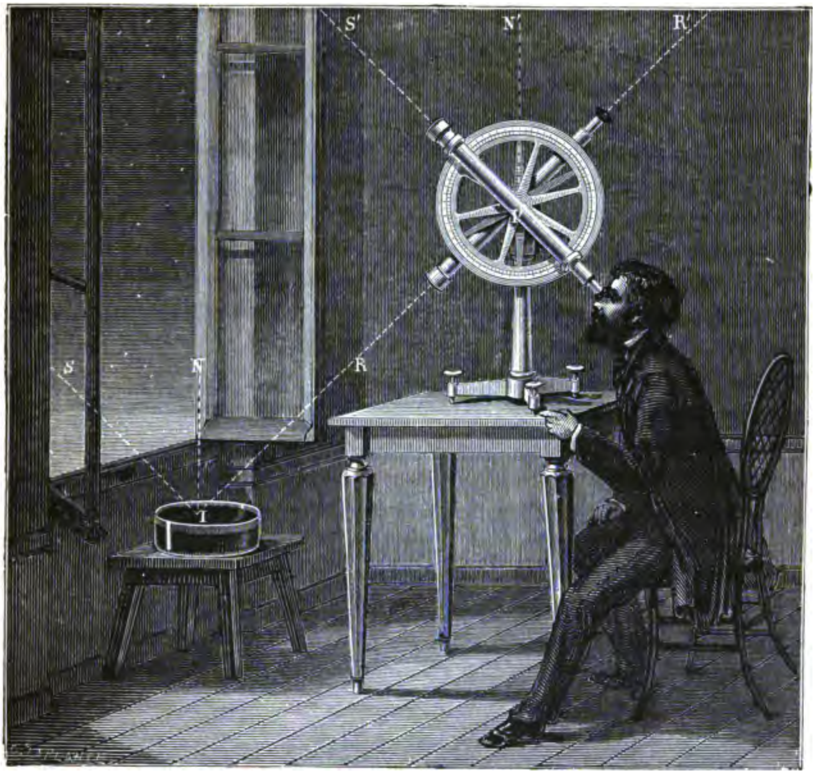


Fig. 661.—Artificial Horizon.

horizon. Observations thus made give even more accurate results than those in which the natural horizon presented by the sea is made the standard of reference.

955. Irregular Reflection.—The reflection which we have thus far been discussing is called *regular reflection.* It is more marked as the reflecting surface is more highly polished, and (except in the case of metals) as the incidence is more oblique. But there is another kind of reflection, in virtue of which bodies, when illuminated, send out light in all directions, and thus become visible. This is called *irregular reflection* or *diffusion.* Regular reflection does not render the reflecting body visible, but exhibits images of surrounding objects. A perfectly reflecting mirror would be itself unseen, and

actual mirrors are only visible in virtue of the small quantity of diffused light which they usually emit. The transformation of incident into diffused light is usually selective; so that, though the incident beam may be white, the diffused light is usually coloured. The power which a body possesses of making such selection constitutes its colour.

The word *reflection* is often used by itself to denote what we have here called *regular reflection*, and we shall generally so employ it.

956. Mirrors.—The mirrors of the ancients were of metal, usually of the compound now known as speculum-metal. Looking-*glasses* date from the twelfth century. They are plates of glass, coated at the back with an amalgam of quicksilver and tin, which forms the reflecting surface. This arrangement has the great advantage of excluding the air, and thus preventing oxidation. It is attended, however, with the disadvantage that the surface of the glass and the surface of the amalgam form two mirrors; and the superposition of the two sets of images produces a confusion which would be intolerable in delicate optical arrangements. The mirrors, or *specula* as they are called, of reflecting telescopes are usually made of *speculum-metal,* which is a bronze composed of about 32 parts of copper to 15 of tin. Lead, antimony, and arsenic are sometimes added. Of late years specula of glass coated in *front* with real silver have been extensively used; they are known as *silvered specula.* A coating of platinum has also been tried, but not with much success. The mirrors employed in optics are usually either *plane* or *spherical.*

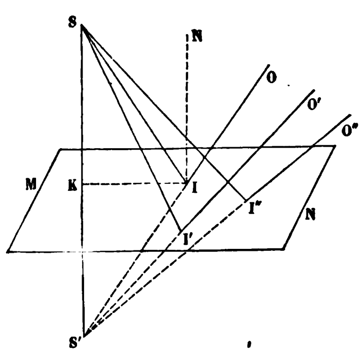


Fig. 662.—Plane Mirror.

957. Plane Mirrors.—By a plane mirror we mean any plane reflecting surface. Its effect, as is well known, is to produce, behind the mirror, images exactly similar, both in form and size, to the real objects in front of it. This phenomenon is easily explained by the laws of reflection.

Let M N (Fig. 662) be a plane mirror, and S a luminous point. Rays S I, S I′, S I″ proceeding from this point give rise to reflected rays I O, I′ O′, I″ O″; and each of these, if produced backwards, will meet the normal S K in a point S′, which is at the same distance behind the mirror that S is in front of

it.[1] The reflected rays have therefore the same directions as if they had come from S′, and the eye receives the same impression as if S′ were a luminous point.

Fig. 663 represents a pencil of rays emitted by the highest point of a candle-flame, and reflected from a plane mirror to the eye of an observer. The reflected rays are divergent (like the incident rays), and if produced backwards would meet in a point, which is the position of the image of the top of the flame.

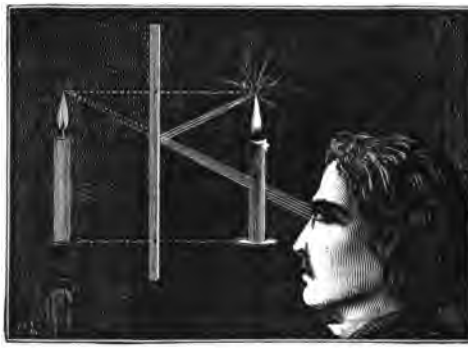


Fig. 663.—Image of Candle.

As an object is made up of points, these principles show that the image of an object formed by a plane mirror must be equal to the object, and symmetrically situated with respect to the plane of the mirror. For example, if AB (Fig. 664) is an object in front of the mirror, an eye placed at O will see the image of the point A at A′, the image of B at B′, and so on for all the other points of the object. The position of the image A′B′ depends only on the positions of the object and of the mirror, and remains stationary as the eye is moved about. It is possible, however, to find positions from which the eye will not see the image at all, the conditions of visibility being the same as if the image were a real object, and the mirror were an opening through which it could be seen.

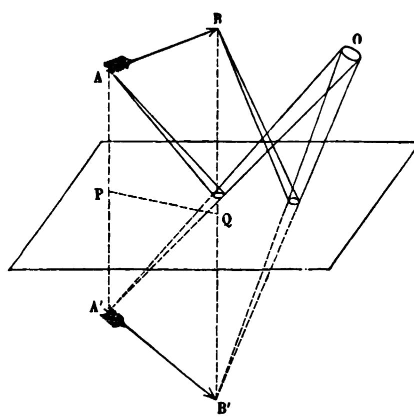


Fig. 664.—Incident and Reflected Pencils.

The images formed by a plane mirror are *erect*. They are not however exact duplicates of the objects from which they are formed,

[1] This is evident from the comparison of the two triangles S K I, S′ K I, bearing in mind that the angle N I S is equal to the alternate angle I S K, and N I O to K S′ I.

but differ from them precisely in the same way as the left foot or hand differs from the right. The image of a printed page is like the appearance of the page as seen through the paper from the back, or like the type from which the page was printed.

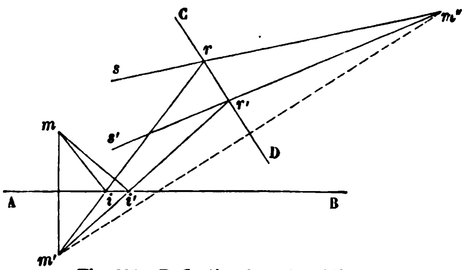


Fig. 665.—Reflection from two Mirrors.

958. Images of Images.—When rays from a luminous point m have been reflected from a mirror A B (Fig. 665), their subsequent course is the same as if they had come from the image m' at the back of the mirror. Hence, if they fall upon a second mirror C D, an image m'' of the first image will be formed at the back of the second mirror. If, after this, they undergo a third reflection, an image of m'' will be formed, and so on indefinitely. The figure shows the actual paths of two rays $m\,i\,r\,s$, $m\,i'\,r'\,s'$. They diverge first from

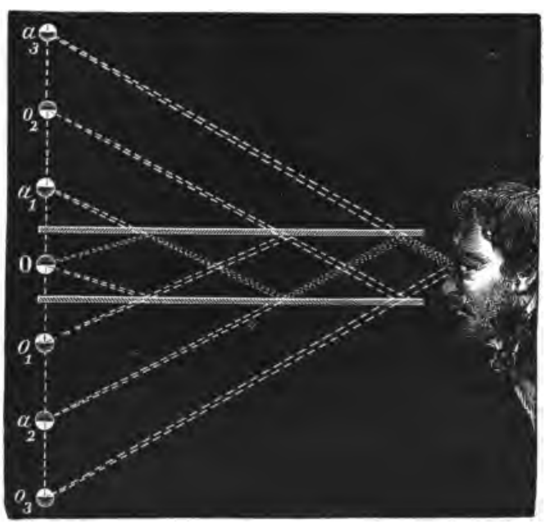


Fig. 666.—Parallel Mirrors.

m, then from m', and lastly from m''. This is the principle of the multiple images formed by two or more mirrors, as in the following experiments.

959. Parallel Mirrors.—Let an object O be placed between two

parallel mirrors which face each other, as in Fig 666. The first reflections will form images $a_1\,o_1$. The second reflections will form images $a_2\,o_2$ of the first images; and the third reflections will form images $a_3\,o_3$ of the second images. The figure represents an eye receiving *the rays which form the third images*, and shows the paths which *these rays* have taken in their whole course from the object O to the eye. The rays by which the same eye sees the other images are omitted, to avoid confusing the figure. A long row of images can thus be seen at once, becoming more and more dim as they recede in the distance, inasmuch as each reflection involves a loss of light.

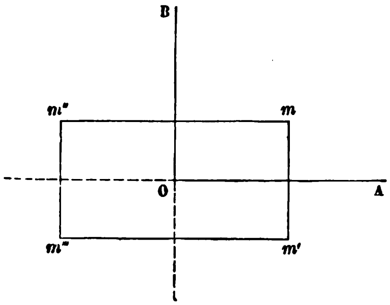


Fig. 667.—Mirrors at Right Angles.

If the mirrors are truly parallel, all the images will be ranged in one straight line, which will be normal to the mirrors. If the mirrors are inclined at any angle, the images will be ranged on the circumference of a circle, whose centre

Fig. 668.—Mirrors at Right Angles.

is on the line in which the reflecting surfaces would intersect if produced. This principle is sometimes employed as a means of adjusting mirrors to exact parallelism.

960. Mirrors at Right Angles.—Let two mirrors O A, O B (Fig. 667),

be set at right angles to each other, facing inwards, and let m be a luminous point placed between them. Images m' m'' will be formed by first reflections, and two coincident images will be formed at m''' by second reflections. No third reflection will occur, for the point m''', being behind the planes of both the mirrors, cannot be reflected in either of them. Counting the two coincident images as one, and also counting the object as one, there will be in all four images, placed at the four corners of a rectangle. Fig. 668 will give an idea of the appearance actually presented when one of the mirrors is vertical and the other horizontal. When both the mirrors are vertical, an observer sees his own image constantly bisected by their common section in a way which appears at first sight very paradoxical.

961. Mirrors Inclined at 60 Degrees.—A symmetrical distribution of images may be obtained by placing a pair of mirrors at any angle which is an aliquot part of 360°. If, for example, they be inclined at 60° to each other, the number of images, counting the object itself as one, will be six. Their position is illustrated by Fig. 669. The object is placed in the sector A C B. The images formed by first reflections are situated in the two neighbouring sectors B C A′, A C B′; the images formed by second reflections are in the sectors B′ C A″, A′ C B″, and these yield, by third reflections, two coincident images in the sector B″ C A″, which is vertically opposite to the sector A C B in which the object lies, and is therefore behind the planes of both mirrors, so that no further reflection can occur.

Fig. 669.—Images in Kaleidoscope.

962. Kaleidoscope.—The symmetrical distribution of images, obtained by two mirrors inclined at an angle which is an aliquot part of four right angles, is the principle of the *kaleidoscope*, an optical toy invented by Sir David Brewster. It consists of a tube containing two glass plates, extending along its whole length, and inclined at an angle of 60°. One end of the tube is closed by a metal plate, with the exception of a hole in the centre, through which the observer looks in; at the other end there are two plates, one of ground and the other of clear glass (the latter being next the eye), with a number of little pieces of coloured glass lying loosely between them. These

coloured objects, together with their images in the mirrors, form symmetrical patterns of great beauty, which can be varied by turning or shaking the tube, so as to cause the pieces of glass to change their positions.

A third reflecting plate is sometimes employed, the cross-section of the three forming an equilateral triangle. As each pair of plates produces a kaleidoscopic pattern, the arrangement is nearly equivalent to a combination of three kaleidoscopes.

The kaleidoscope is capable of rendering important aid to designers.

Fig. 670.—Kaleidoscopic Pattern.

Fig. 670 represents a pattern produced by the equilateral arrangement of three reflectors just described.

963. Pepper's Ghost.—Many ingenious illusions have been contrived, depending on the laws of reflection from plane surfaces. We shall mention two of the most modern.

In the *magic cabinet*, there are two vertical mirrors hinged at the two back corners of the cabinet, and meeting each other at a right angle, so as to make angles of 45° with the sides, and also with the back. A spectator seeing the images of the two sides, mistakes them for the back, which they precisely resemble; and performers may be concealed behind the mirrors when the cabinet appears empty. If one of the persons thus concealed raises his head above the mirrors, it will appear to be suspended in mid-air without a body.

The striking spectral illusion known as *Pepper's Ghost* is produced by reflection from a large sheet of unsilvered glass, which is so arranged that the actors on the stage are seen through it, while other actors, placed in strong illumination, and out of the direct view of the spectators, are seen by reflection in it, and appear as ghosts on the stage.

964. Deviation produced by Rotation of Mirror.—Let A B (Fig. 671) represent a mirror perpendicular to the plane of the paper, and capable of being rotated about an axis through C, also perpendicular to the paper; and let I C represent an incident ray. When the mirror is in the position A B, perpendicular to I C, the ray will be reflected directly back upon its course; but when the mirror is turned through the acute angle A C A′, the reflected ray will take the direction C R, making with the normal C N an angle N C R, equal to the angle of incidence N C I. The deviation I C R of the reflected ray, produced by rotating the mirror, is therefore double of the angle I C N or A C A′, through which the mirror has been turned; and if, starting from the position A′B′, we turn the mirror through a further angle θ, the reflected ray C R will be turned through a further angle 2θ. It thus appears, that, *when a plane mirror is rotated in the plane of incidence, the direction of the reflected ray is changed by double the angle through which the mirror is turned.* Conversely, if we assign a constant direction C I to the reflected ray, the direction of the incident ray R C must vary by double the angle through which the mirror is turned.

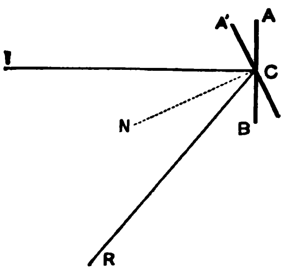


Fig. 671.—Effect of rotating a Mirror.

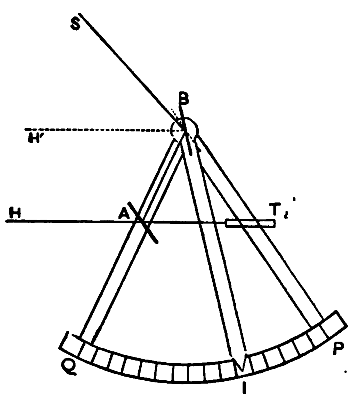


Fig. 672.—Sextant.

965. Hadley's Sextant.—The above principle is illustrated in the nautical instrument called the *sextant* or *quadrant*, which was invented by Newton, and reinvented by Hadley. It serves for measuring the angle between any two distant objects as seen from the station occupied by the observer. Its essential parts are represented in Fig. 672.

It has two plane mirrors A, B, one of which, A, is fixed to the frame of the instrument, and is only partially silvered, so that a distant object in the direction A H can be seen through the unsilvered part. The other mirror B is mounted on a movable arm B I, which carries an index I, traversing a graduated arc P Q. When the two mirrors are parallel, the index is at P, the zero of the graduations, and a ray H′ B incident on B parallel to H A, will be reflected first along B A, and then along A T, the continuation of H A. The observer looking through the telescope T thus sees, by two reflections, the same objects which he also sees directly through the unsilvered part of the mirror. Now let the index be advanced through an angle θ; then, by the principles of last section, the incident ray S B makes with H′ B, or H A, an angle 2θ. The angle between S B and H A would therefore be given by reading off the angle through which the index has been advanced, and doubling; but in practice the arc P Q is always graduated on the principle of marking half degrees as whole ones, so that the reading at I is the required angle 2θ. In using the instrument, the two objects which are to be observed are brought into apparent coincidence, one of them being seen directly, and the other by successive reflection from the two mirrors. This coincidence is not disturbed by the motion of the ship; but unpractised observers often find a difficulty in keeping both objects in the field of view. Dark glasses, not shown in the figure, are provided for protecting the eye in observations of the sun, and a vernier and reading microscope are provided instead of the pointer I.

966. Spherical Mirrors.—By a spherical mirror is meant a mirror

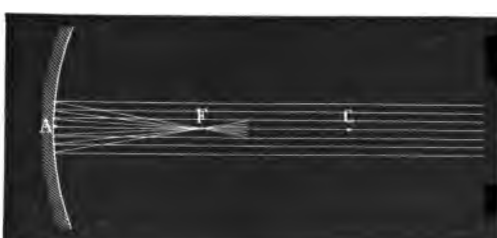


Fig. 673.—Principal Focus.

whose reflecting surface is a portion (usually a very small portion) of the surface of a sphere. It is concave or convex according as the inside or outside of the spherical surface yields the reflection. The centre of the sphere (C, Fig. 673) is called the *centre of curvature* of

the mirror. If the mirror has a circular boundary, as is usually the case, the central point A of the reflecting surface may conveniently be called the *pole* of the mirror. *Centre of the mirror* is an ambiguous phrase, being employed sometimes to denote the pole, and sometimes the centre of curvature. The line A C is called the *principal axis* of the mirror, and any other straight line through C which meets the mirror is called a *secondary axis*.

When the incident rays are parallel to the principal axis, the reflected rays converge to a point F, which is called the *principal focus*. This law is rigorously true for parabolic mirrors (generated by the revolution of a parabola about its principal axis). For spherical mirrors it is only approximately true, but the approximation is very close if the mirror is only a very small portion of an entire sphere. In grinding and polishing the specula of large reflecting telescopes, the attempt is made to give them, as nearly as possible, the parabolic form. Parabolic mirrors are also frequently employed

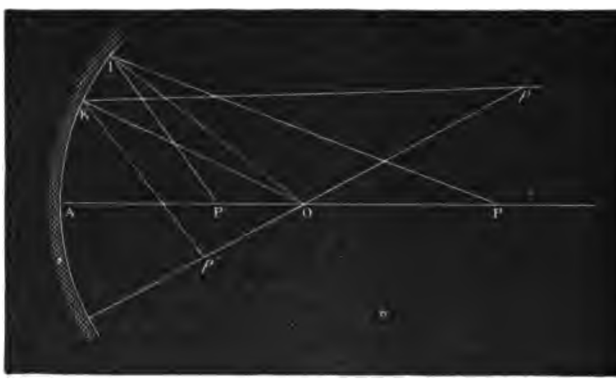


Fig. 674.—Theory of Conjugate Foci.

to reflect, in a definite direction, the rays of a lamp placed at the focus.

Rays reflected from the circumferential portion of a spherical mirror are always too convergent to concur exactly with those reflected from the central portion. This deviation from exact concurrence is called *spherical aberration*.

967. **Conjugate Foci.**—Let P (Fig. 674) be a luminous point situated on the principal axis of a spherical mirror, and let P I be one of the rays which it sends to the mirror. Draw the normal O I, which is simply a radius of the sphere. Then O I P is the angle of incid-

ence, and the angle of reflection O I P′ must be equal to it; hence O I bisects an angle of the triangle P I P′, and therefore we have

$$\frac{IP}{IP'} = \frac{OP}{OP'}$$

Let p, p' denote A P, A P′ respectively, and let r denote the radius of the sphere. Then, if the angular aperture of the mirror is small, I P is sensibly equal to p, and I P′ to p'. Substituting these approximate values, the preceding equation becomes

$$\frac{p}{p'} = \frac{p-r}{r-p'}; \text{ whence } pr + p'r = 2pp';$$

or, dividing by $pp'r$,

$$\frac{1}{p} + \frac{1}{p'} = \frac{2}{r}. \qquad (a)$$

This formula determines the position of the point P′, in which the reflected ray cuts the principal axis, and shows that it is, to the accuracy of our approximation, independent of the position of the point I; that is to say, all the rays which P sends to the mirror are reflected to the same point P′. We have assumed P to be on the principal axis. If we had taken it on a secondary axis, as at p (Fig. 674), we should have found, by the same process of reasoning, that the reflected rays would all meet in a point p' on that secondary axis. The distinction between primary and secondary axes, in the case of a spherical mirror, is in fact merely a matter of convenience, not representing any essential difference of property. Hence we can lay down the following general proposition as true within limits of error corresponding to the approximate equalities which we have above assumed as exact:—

Rays proceeding from any given point in front of a concave spherical mirror, are reflected so as to meet in another point; and the line joining the two points passes through the centre of the sphere.

It is evident that rays proceeding from the second point to the mirror, would be reflected to the first. The relation between them is therefore mutual, and they are hence called *conjugate foci.* By a *focus* in general is meant a point in which a number of rays, which originally came from the same point, meet (or would meet if produced); and the rays which thus meet, taken collectively, are called a *pencil.* Fig. 675 represents two pencils of rays whose foci S *s* are conjugate, so that, if either of them be regarded as an incident pencil, the other will be the corresponding reflected pencil.

We can now explain the formation of images by concave mirrors. Each point of the object sends a pencil of rays to the mirror, which converge, after reflection, to the conjugate focus. If the eye of the observer be placed beyond this point of concourse, and in the path of the rays, they will present to him the same appearance as if they

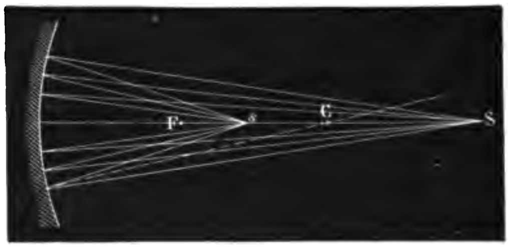


Fig. 675.—Conjugate Foci.

had come from this point as origin. The image is thus composed of points which are the conjugate foci of the several points of the object.

968. Principal Focus.—If, in formula (a) of last section, we make p increase continually, the term $\frac{1}{p}$ will continually decrease, and will vanish as p becomes infinite. This is the case of rays parallel to the principal axis, for parallel rays may be regarded as coming from a point at infinite distance. The formula then becomes

$$\frac{1}{p'} = \frac{2}{r}; \text{ whence } p' = \frac{r}{2};$$

that is to say, *the principal focal distance is half the radius of curvature.* This distance is often called the *focal length* of the mirror. If we denote it by f, the general formula becomes

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{f}. \qquad (b)$$

969. Discussion of the Formula.—By the aid of this formula we can easily trace the corresponding movements of conjugate foci.

If p is positive and very large, p' is a very little greater than f; that is to say, the conjugate focus is a very little beyond the principal focus.

As p diminishes, p' increases, until they become equal, in which case each of them is equal to r or $2f$; that is to say, the conjugate foci move towards each other till they coincide at the centre of curvature. This last result is obvious in itself; for rays from the centre

of curvature are normal to the mirror, and are therefore reflected directly back.

As p continues to diminish, the two foci, as it were, change places; the luminous point advancing from the centre of curvature to the principal focus, while the conjugate focus moves away from the centre of curvature to infinity.

As the luminous point continues to approach the mirror, $\frac{1}{p}$ is greater than $\frac{1}{f}$, and hence $\frac{1}{p'}$, and therefore also p', must be negative. The physical interpretation of this result is that the conjugate focus is *behind* the mirror, as at s (Fig. 676), and that the reflected

Fig. 676.—Virtual Focus.

rays diverge as if they had come from this point. Such a focus is called *virtual*, while a focus in which rays actually meet is called *real*. As the luminous point moves up from F to the mirror, the conjugate focus moves up from an infinite distance at the back, and meets it at the surface of the mirror.

If S is a real luminous point sending rays to the mirror, it must of necessity lie in front of the mirror, and p therefore cannot be negative; but when we are considering images of images this restriction no longer holds. If an incident beam, for example, converges towards a point s at the back of the mirror, it will be reflected to a point S in front. In this case p is negative, and p' positive. The conjugate foci S s have in fact changed places.

It appears from the above investigation that there are two principal cases, as regards the positions of conjugate foci of a concave mirror.

1. One focus between F and C; and the other beyond C.

2. One focus between F and the mirror; and the other behind the mirror.

In the former case, the foci move to meet each other at C; in the latter, they move to meet each other at the surface of the mirror.

970. Formation of Images.—We are now in a position to discuss the formation of images by concave mirrors. Let A B (Fig. 677) be an object placed in front of a concave mirror, at a distance greater than its radius of curvature. All the rays which diverge from A will be reflected to the conjugate focus a. Hence this point can be found by the following construction. Draw through A the ray A A′ parallel to the principal axis, and draw its path after reflection, which must of necessity pass through the principal focus. The intersection of this reflected ray with the secondary axis through A will be the point required. A similar construction will give the conjugate focus

Fig. 677.—Formation of Image.

corresponding to any other point of the object; b, for example,[1] is the focus conjugate to B. Points of the object lying between A and B will have their conjugate foci between a and b. An eye placed behind the object A B will accordingly receive the same impression from the reflected rays as if the image $a\,b$ were a real object.

Since the lines joining corresponding points of object and image cross at the point C, which lies between them when the image is real, a real image formed by a concave mirror is always inverted.

971. Size of Image.—As regards the comparative sizes of object and image, it is obvious, from similar triangles, that their linear dimensions are *directly as their distances from* C, *the centre of curvature.*

Again, we have proved in § 967 that, in the notation of that section,

$$\frac{\mathrm{IP}}{\mathrm{IP'}} = \frac{\mathrm{OP}}{\mathrm{OP'}};$$

[1] It is only by accident that b happens to lie on A A′ in the figure.

or, the distances of object and image from the mirror are directly as their distances from the centre of curvature. Their linear dimensions are therefore *directly as their distances from the mirror.*

Again, by equation (*b*),

$$\frac{1}{p'} = \frac{1}{f} - \frac{1}{p} = \frac{p-f}{fp},$$

whence

$$\frac{p}{p'} = \frac{p-f}{f}, \qquad (c)$$

where $p-f$ is the distance of the object from the principal focus. Hence the linear dimensions of object and image are *in the ratio of the distance of the object from the principal focus to the focal length.*

These three rules are perfectly general, both for concave and convex mirrors.

The first rule shows that the object and image are equal when

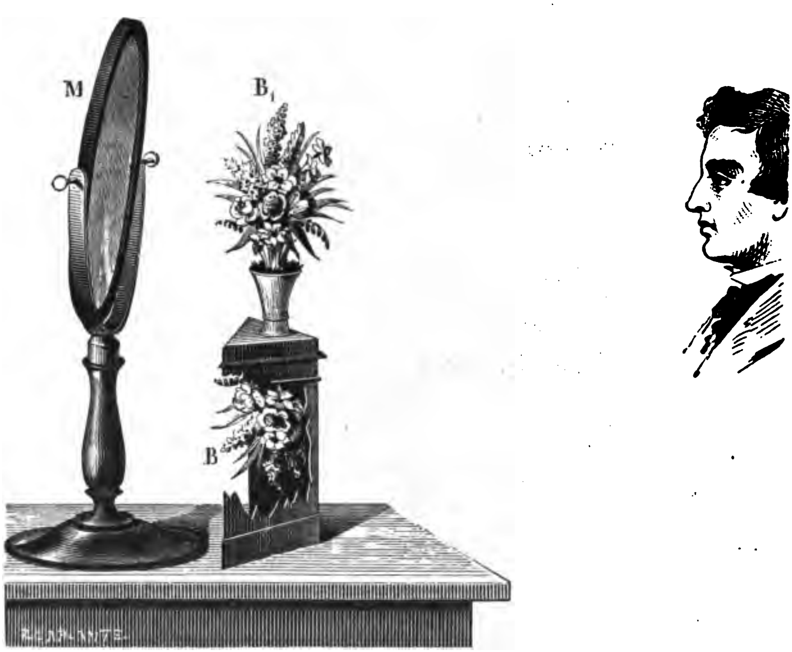


Fig. 678.—Experiment of Phantom Bouquet.

they coincide at the reflecting surface, and that, as they separate from this point in opposite directions, that which moves away from the centre of curvature continually gains in size upon the other.

The second rule shows that the object and image are equal when

they coincide at the centre of curvature, and that as they separate from this point, in opposite directions, that which moves away from the mirror continually gains in size upon the other.

The third rule shows that, when the object is at the principal focus, the size of the image is infinite.

972. Experiment of the Phantom Bouquet.—Let a box, open on one side, be placed in front of a concave mirror (Fig. 678), at a distance about equal to its radius of curvature, and let an inverted bouquet be suspended within it, the open side of the box being next the mirror. By giving a proper inclination to the mirror, an image of the bouquet will be obtained in mid-air, just above the top of the box. As the bouquet is inverted, its image is erect, and a real vase may be placed in such a position that the phantom bouquet shall appear to be standing in it. The spectator must be full in front of the mirror, and at a sufficient distance for all parts of the image to lie between his eyes and the mirror. When the colours of the bouquet are bright, the image is generally bright enough to render the illusion very complete.

973. Images on a Screen.—Such experiments as that just described

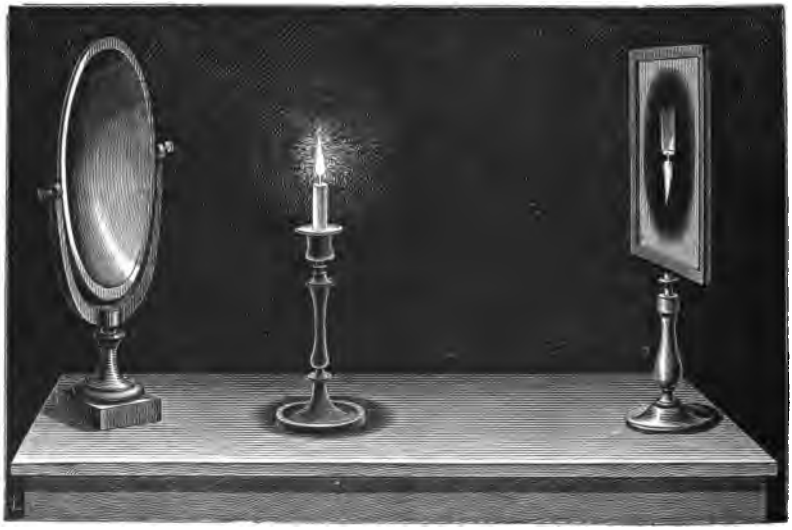


Fig. 679.—Image on Screen.

can only be seen by a few persons at once, since they require the spectator to be in a line with the image and the mirror. When an image is projected on a screen, it can be seen by a whole audience

at once, if the room be darkened and the image be large and bright. Let a lighted candle, for example, be placed in front of a concave mirror, at a distance exceeding the focal length, and let a screen be placed at the conjugate focus; an inverted image of the candle will be depicted on the screen. Fig. 679 represents the case in which the candle is at a distance less than the radius of curvature, and the image is accordingly magnified.

By this mode of operating, the formula for conjugate focal distances can be experimentally verified with considerable rigour, care being taken, in each experiment, to place the screen in the position which gives the most sharply defined image.

974. Difference between Image on Screen, and Image as seen in Mid-air. Caustics.—For the sake of simplicity we have made some statements regarding visible images which are not quite accurate; and we must now indicate the necessary corrections.

Images thrown on a screen have a determinate position, and are really the loci of the conjugate foci of the points of the object; but

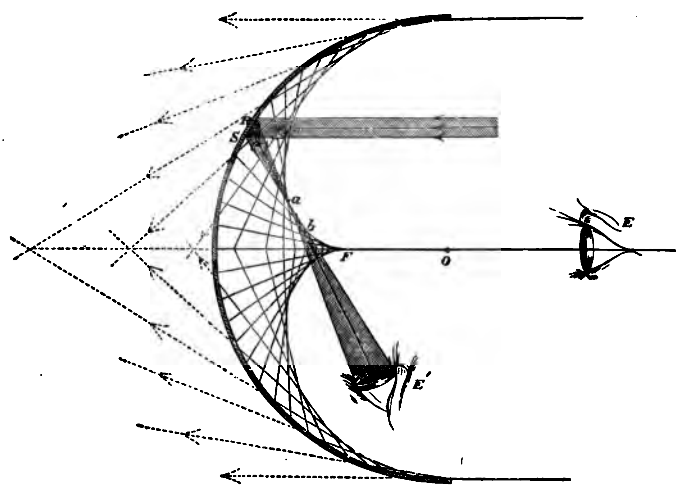


Fig. 680.—Position of Image in Oblique Reflection.

this is not rigorously true of images seen directly. They change their position to some extent, according to the position of the observer.

The actual state of things is explained by Fig. 680. The plane of the figure[1] is a principal plane (that is, a plane containing the principal axis) of a concave hemispherical mirror, and the incident rays

[1] Figs. 680 and 698 are borrowed, by permission, from Mr. Osmund Airy's *Geometrical Optics*.

are parallel to the principal axis. All the rays reflected in the plane of the figure touch a certain curve called a *caustic curve*, which has a cusp at F, the principal focus; and the direction in which the image is seen by an eye situated in the plane of the figure is determined by drawing from the eye a tangent to this caustic. If the eye be at E, on the principal axis, the point of contact will be F; but when the rays are received obliquely, as at E′, it will be at a point a not lying in the direction of F. For an eye thus situated, a is called the *primary focus*, and the point where the tangent at a cuts the principal axis is called the *secondary focus*. When the eye is moved in the plane of the diagram, the apparent position of the image (as determined by its remaining in coincidence with a cross of threads or other mark) is the primary focus; and when the eye is moved perpendicular to the plane of the diagram, the apparent position of the image is the secondary focus.[1] If we suppose the diagram to rotate about the principal axis, it will still remain true in all positions, and the surface generated by this revolution of the caustic curve is the *caustic surface*. Its form and position vary with the position of the point from which the incident rays proceed; and it has a cusp at the focus conjugate to this point.

There is always more or less blurring, in the case of images seen obliquely (except in plane mirrors), by reason of the fact that the point of contact with the caustic surface is not the same for rays entering different parts of the pupil of the eye.

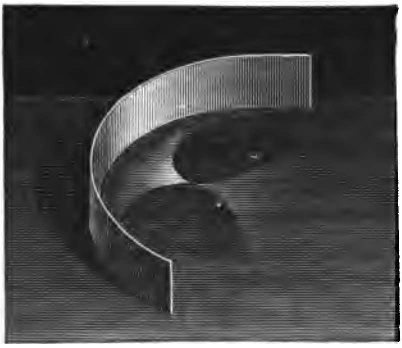


Fig. 681.—Caustic by Reflection.

A caustic curve can be exhibited experimentally by allowing the rays of the sun or of a lamp to fall on the concave surface of a strip of polished metal bent into the form of a circular arc, as in Fig. 681, the reflected light being received on a sheet of white paper on which the strip rests. The same effect may often be observed on the surface of a cup of tea, the reflector in this case being the inside of the tea-cup.

[1] Since every ray incident parallel to the principal axis, is reflected through the principal axis. If the incident rays diverged from a point on the principal axis, they would still be reflected through the principal axis.

The image of a luminous point received upon a screen is formed by all the rays which touch the corresponding caustic surface. The brightest and most distinct image will be formed at the cusp, which is, in fact, the conjugate focus; but there will be a border of fainter light surrounding it. This source of indistinctness in images is an example of *spherical aberration* (§ 967).

975. Image on a Screen by Oblique Reflection.—If we attempt to throw upon a screen the image of a luminous point by means of a concave mirror very oblique to the incident rays, we shall find that no image can be obtained at all resembling a point; but that there are two positions of the screen in which the image becomes a line.

In the annexed figure (Fig. 682), which represents on a larger scale a portion of Fig. 680, *a c*, *b d* are rays from the highest and lowest points of the portion R S of the hemispherical mirror, which portion we suppose to be small in both its dimensions in comparison with the radius of curvature; and we may suppose the rest of the hemisphere to be removed, so that R S will represent a small concave mirror receiving a pencil very obliquely.

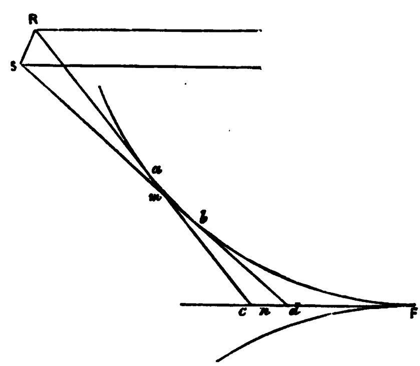


Fig. 682.—Formation of Focal Lines.

Then, if a screen be held perpendicular to the plane of the diagram, at *m*, where the section of the pencil by the plane of the diagram is narrowest, a blurred line of light will be formed upon it, the length of the line being perpendicular to the plane of the diagram. This is called the *primary focal line.*

The *secondary focal line* is *c d*, which, if produced, passes through the centre of curvature of the mirror, and also through the point from which the incident light proceeds. This line is very sharply formed upon a screen held so as to coincide with *c d* and to be perpendicular to the plane of the diagram. Its edges are much better defined than those of the primary line; and its position in space is also more definite. If the mirror is used as a burning-glass to collect

the sun's rays, ignition will be more easily obtained at one of these lines than in any intermediate position.

Focal lines can also be seen directly. In this case a small element of the mirror sends all its reflected rays to the eye, the rays from opposite sides of the element crossing each other at the focal lines, before they reach the eye. It is possible, in certain positions of the eye, to see either focal line at pleasure, by altering the focal adjustment of the eye; or the two may be seen with imperfect definition crossing each other at right angles. The experiment is easily made by employing a gas flame, turned very low, as the source of light. One line is in the plane of incidence, and the other is normal to this plane.

976. Virtual Image in Concave Mirror.—Let an object be placed, as in Figs. 683, 684, in front of a concave mirror, at a distance less than that of the principal focus. The rays incident on the mirror from any point of it, as A (Fig. 683), will be reflected as a divergent pencil, the focus from which they diverge being a point b at the back of the mirror. To find this point, we may trace the course of a ray through A parallel to the principal axis. Such a ray will be reflected to the principal focus F, and by producing this reflected ray backwards till it meets the secondary axis C A, the point b, which is the conjugate focus of A, is determined. We can find in the same way the position of a, the conjugate focus of B, and it is obvious that the image of A B will be erect and magnified.

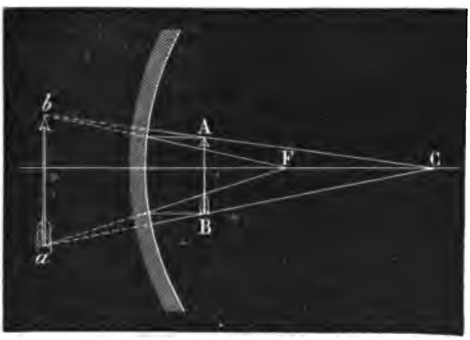


Fig. 683.—Formation of Virtual Image.

977. Remarks on Virtual Images.—A virtual image cannot be projected on a screen; for the rays which produce it do not actually pass through its place, but only seem to do so. A screen placed at $a\,b$ would obviously receive none of the reflected light whatever.

[1] The "elongated figure of 8" which is often mentioned in connection with the secondary focal line, is obtained by turning the screen about n the middle point of $c\,d$, so as to blur both ends of the image by bad focussing. It will be observed, from an inspection of the diagram, that $c\,d$ is very oblique to the reflected rays.

If we neglect the blurring of the primary line, we may describe the part of the pencil lying between the two lines as a tetrahedron, of which the two lines are opposite edges.

The images seen in a plane mirror are virtual; and any spherical mirror, whether concave or convex, is nearly equivalent to a plane mirror, when the distance of the object from its surface is small in comparison with the radius of curvature.

978. Convex Mirrors.—It is easily shown, by a simple construction, that rays incident from any luminous point upon a convex mirror, diverge after reflection. The principal focus, and the foci conjugate to all points external to the sphere, are therefore virtual.

Fig. 684.—Virtual Image in Concave Mirror.

To adapt formulæ (a) and (b) of the preceding sections to the case of convex mirrors, we have only to alter the sign of the term $\frac{2}{r}$ or $\frac{1}{f}$; so that for a convex mirror we shall have

$$\frac{1}{p}+\frac{1}{p'}=-\frac{1}{f}=-\frac{2}{r}; \qquad (c)$$

r and f being here regarded as essentially positive.

From this formula it is obvious that one at least of the two distances p, p' must be negative; that is to say, one at least of any pair of conjugate foci must lie behind the mirror.

The construction for an image (Fig. 685) is the same as in the case of concave mirrors. Through any selected point of the object draw a ray parallel to the principal axis; the reflected ray, if produced backwards, must pass through the principal focus, and its intersection with the secondary axis through the selected point determines the corresponding point of the image. The image of an external object will evidently be erect, and smaller than the object. Repeating the same construction when the object is nearer to the mirror, we see that the image will be larger than before.

The linear dimensions of an object and its image, whether in the case of a convex or a concave mirror, are directly proportional to

their distances from the centre of curvature, and are also directly proportional to their distances from the mirror. The image is inverted or erect according as the centre of curvature does or does not

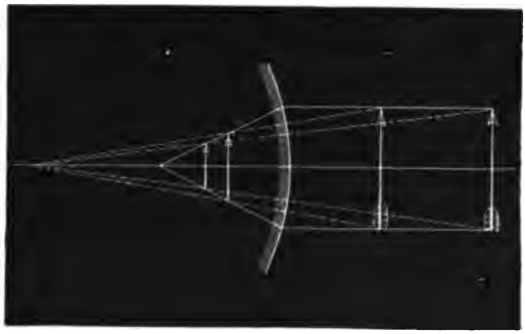


Fig. 685.—Formation of Image in Convex Mirror.

lie between the object and its image. In the case of a convex mirror the centre never lies between them (if the object be real), and therefore the image is always erect.

Convex mirrors are very seldom employed in optical instruments.

The silvered globes which are frequently used as ornaments, are examples of convex mirrors, and present to the observer at one view an image of nearly the whole surrounding landscape. As the part of the mirror in which he sees this image is nearly an entire hemisphere, the deformation of the image is very notable, straight lines being reflected as curves.

979. Anamorphosis.—Much greater deformations are produced by cylindric mirrors. A cylindric mirror, when the axis of the cylinder is vertical, behaves like a plane mirror as regards the angular magnitude under which the height of the image is seen, and like a spherical mirror as regards the breadth of the image. If it be a convex cylinder, it causes bodies to appear unduly contracted horizontally in proportion to their heights. Distorted pictures are sometimes drawn upon paper, according to such a system that when they are seen reflected in a cylindric mirror properly placed, as in Fig. 686, the distortion is corrected, and while the picture appears a mass of confusion, the image is instantly recognized. This restoration of true proportion in a picture is called *anamorphosis*.

980. Medical Applications.—Concave mirrors are frequently used

to concentrate light upon an object for the purpose of rendering it more distinctly visible.

The *ophthalmoscope* is a small concave mirror, with a small hole in its centre, through which the observer looks from behind, while he

Fig. 686.—Anamorphosis.

directs a beam of reflected light from a lamp into the pupil of the patient's eye. In this way (with the help sometimes of a lens) the retina can be rendered visible, and can be minutely examined.

The *laryngoscope* consists of two mirrors. One is a small plane mirror, with a handle attached, at an angle of about 45° to its plane. This small mirror is held at the back of the patient's mouth, so that the observer, looking into it, is able by reflection to see down the patient's throat, the necessary illumination being supplied by a concave mirror, strapped to the observer's forehead, by means of which the light from a lamp is reflected upon the plane mirror, which again reflects it down the throat.

Some additions to this chapter will be found at page 1086.

CHAPTER LXIX.

REFRACTION.

981. Refraction.—When a ray of light passes from one transparent medium to another, it undergoes a change of direction at the surface of separation, so that its course in the second medium makes an angle with its course in the first. This changing of direction is called *refraction.*

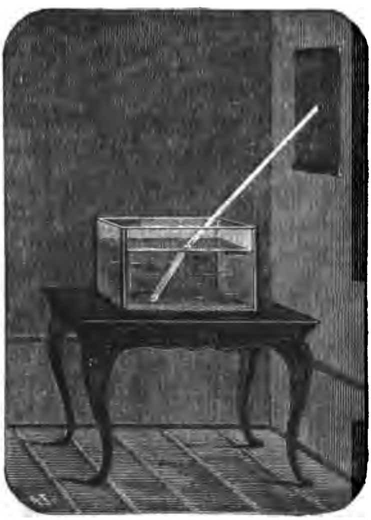


Fig. 687.—Refraction.

The phenomenon can be exhibited by admitting a beam of the sun's rays into a dark room, and receiving it on the surface of water contained in a rectangular glass vessel (Fig. 687). The path of the beam will be easily traced by its illumination of the small solid particles which lie in its course.

The following experiment is a well-known illustration of refraction:—A coin *m n* (Fig. 688) is laid at the bottom of a vessel with opaque sides, and a spectator places himself so that the coin is just hidden from him by the side of the vessel; that is to say, so that the line *m* A in the figure passes just above his eye. Let water now be poured into the vessel, care being taken not to displace the coin. The bottom of the vessel will appear to rise, and the coin will come into sight. Hence a pencil of rays from *m* must have entered the spectator's eye. The pencil in fact undergoes a sudden bend at the surface of the water, and thus reaches the eye by a crooked course,

in which the obstacle A is evaded. If the part of the pencil in air be produced backwards, its rays will approximately meet in a point m', which is therefore the image of m. Its position is not correctly indicated in the figure, being placed too much to the left (§ 990).

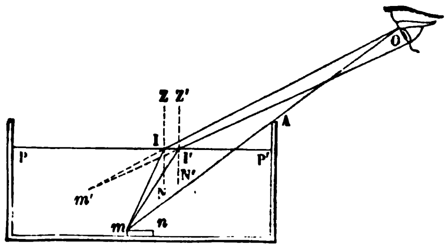


Fig. 688.—Experiment of Coin in Basin.

The broken appearance presented by a stick (Fig. 689) when partly immersed in water in an oblique position, is similarly explained, the part beneath the water being lifted up by refraction.

982. Refractive Powers of Different Media.—In the experiments of the coin and stick, the rays, in leaving the water, are bent away from the normals Z I N, Z′ I′ N′ at the points of emergence; in the experiment first described (Fig. 687), on the other hand, the rays, in passing from air into water, are bent nearer to the normal. In every case the path which the rays pursue in going is the same as they would pursue in returning; and of the two media concerned, that in which the ray makes the smaller angle with the normal is said to have greater refractive power than the other, or to be more highly refracting.

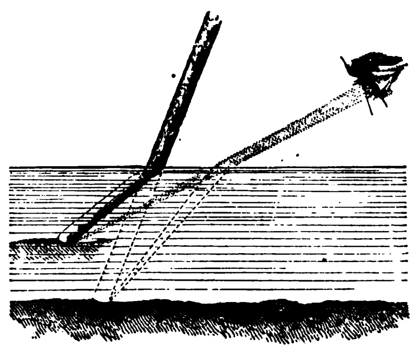


Fig. 689.—Appearance of Stick in Water.

Liquids have greater refractive power than gases, and as a general rule (subject to some exceptions in the comparison of dissimilar substances) the denser of two substances has the greater refracting power. Hence it has become customary, in enunciating some of the laws of optics, to speak of the *denser* medium and the *rarer* medium, when the more correct designations would be *more refractive* and *less refractive*.

983. Laws of Refraction.—The quantitative law of refraction was not discovered till quite modern times. It was first stated by Snell, a Dutch philosopher, and was made more generally known by Descartes, who has often been called its discoverer.

Let R I (Fig. 690) be a ray incident at I on the surface of separation of two media, and let I S be the course of the ray after refraction. Then the angles which R I and I S make with the normal are called the *angle of incidence* and the *angle of refraction* respectively; and the first law of refraction is that these angles lie in the same plane, or *the plane of refraction is the same as the plane of incidence.*

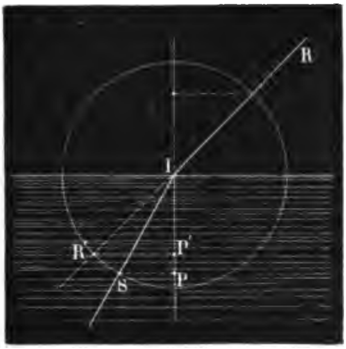


Fig. 690.—Law of Refraction.

The law which connects the magnitudes of these two angles, and which was discovered by Snell, can only be stated either by reference to a geometrical construction, or by employing the language of trigonometry. Describe a circle about the point of incidence I as centre, and drop perpendiculars, from the points where it cuts the rays, on the normal. The law is that these perpendiculars R′ P′, S P, will have a constant ratio; or *the sines of the angles of incidence and refraction are in a constant ratio.* It is often referred to as the *law of sines.*

The angle by which a ray is turned out of its original course in undergoing refraction is called its *deviation.* It is zero if the incident ray is normal, and always increases with the angle of incidence.

984. Verification of the Law of Sines.—These laws can be verified by means of the apparatus represented in Fig. 691, which is very similar to that employed by Descartes. It has a vertical divided circle, to the front of which is attached a cylindrical vessel, half-filled with water or some other transparent liquid. The surface of the liquid must pass exactly through the centre of the circle. I is a movable mirror for directing a reflected beam of solar light on the centre O. The beam must be directed centrally through a short tube attached to the mirror, and to facilitate this adjustment the tube is furnished with a diaphragm with a hole in its centre. The arm O *a* is movable about the centre of the circle, and carries a vernier for measuring the angle of incidence. The ray undergoes refraction at O; and the angle of refraction is measured by means of a second arm O R, which is to be moved into such a position that the diaphragm of its tube receives the beam centrally. No refraction

occurs at emergence, since the emergent beam is normal to the surfaces of the liquid and glass; the position of the arm accordingly indicates the direction of the refracted ray. The angles of incidence and refraction can be read off at the verniers carried by the two arms; and the ratio of their sines will be found constant. The sines can also be directly measured by employing sliding-scales as indicated in the figure, the readings being taken at the extremity of each arm.

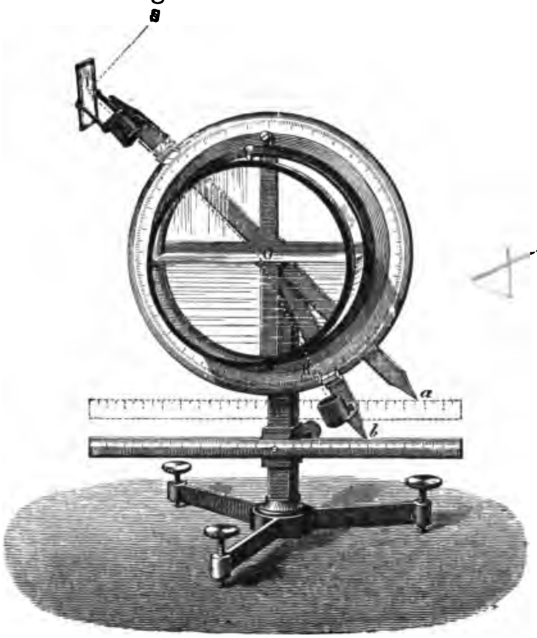


Fig. 691.—Apparatus for Verifying the Law.

It would be easy to make a beam of light enter at the lower side of the apparatus, in a radial direction; and it would be found that the ratio of the sines was precisely the same as when the light entered from above. This is merely an instance of the general law, that the course of a returning ray is the same as that of a direct ray.

985. Airy's Apparatus.—The following apparatus for the same purpose was invented, many years ago, by the present astronomer royal. B′ is a slider travelling up and down a vertical stem. A C′ and B C are two rods pivoted on a fixed point B of the vertical stem. C′ B′ and C B′ are two other rods jointed to the former at C′ and C, and pivoted at their lower ends on the centre

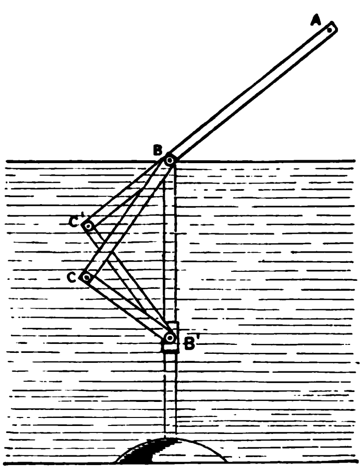


Fig. 692.—Airy's Apparatus.

of the slider. B C is equal to B′ C′, and B C′ to B′ C. Hence the two triangles B C B′, B′ C′ B are equal to one another in all positions of the slider, their common side B B′ being variable, while the other two sides of each remain unchanged in length though altered in position.

The ratio $\frac{BC}{CB'}$ or $\frac{B'C'}{C'B}$ is made equal to the index of refraction of the liquid in which the observation is to be made. For water this ratio will be $\frac{4}{3}$. Then, if the apparatus is surrounded with water up to the level of B, A B C will be the path of a ray, and a stud at C will appear in the same line with studs at A and B; for we have

$$\frac{\sin C'BB'}{\sin CBB'} = \frac{\sin C'BB'}{\sin C'B'B} = \frac{C'B'}{C'B} = \frac{4}{3}.$$

986. Indices of Refraction.—The ratio of the sine of the angle of incidence to the sine of the angle of refraction when a ray passes from one medium into another, is called the *relative index of refraction* from the former medium to the latter. When a ray passes from vacuum into any medium this ratio is always greater than unity, and is called the *absolute index of refraction*, or simply the *index of refraction*, for the medium in question. The relative index of refraction from any medium A into another B is always equal to the absolute index of B divided by the absolute index of A. The absolute index of air is so small that it may usually be neglected in comparison with those of solids and liquids: but strictly speaking, the relative index for a ray passing from air into a given substance must be multiplied by the absolute index for air, in order to obtain the absolute index of refraction for the substance.

The following table gives the indices of refraction of several substances:—

INDICES OF REFRACTION.[1]

Diamond,	2·44 to 2·755	Alcohol,	1·372
Sapphire,	1·794	Aqueous humour of eye,	1·337
Flint-glass,	1·576 to 1·642	Vitreous humour,	1·339
Crown-glass,	1·531 to 1·563	Crystalline lens, outer coat,	1·337
Rock-salt,	1·545	" " under coat,	1·379
Canada balsam,	1·540	" " central portion,	1·400
Bisulphide of carbon,	1·678	Sea water,	1·343
Linseed oil (sp. gr. ·932),	1·482	Pure water,	1·336
Oil of turpentine (sp. gr. ·885),	1·478	Air at 0° C. and 760 mm	1·000294

987. Critical Angle.—We see, from the law of sines, that when the

[1] The index of refraction is always greater for violet than for red (see Chap. lxxii.). The numbers in this table are to be understood as mean values.

incident ray is in the less refractive of the two media, to every possible angle of incidence there is a corresponding angle of refraction. This, however, is not the case when the incident ray is in the more refractive of the two media. Let S O, S′ O, S″ O (Fig. 693) be inci-

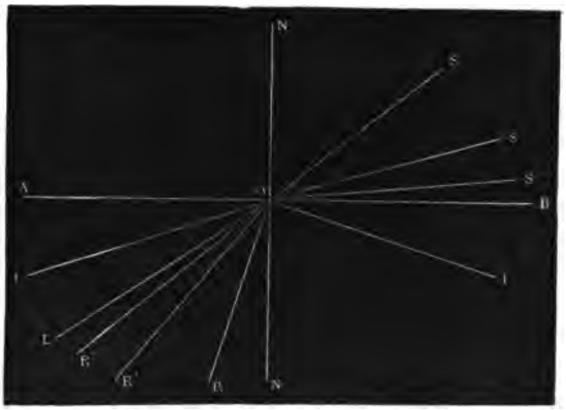


Fig. 693.—Critical Angle.

dent rays in the less refractive medium, and O R, O R′, O R″ the corresponding refracted rays. There will be a particular direction of refraction O L corresponding to the angle of incidence of 90°. Conversely, incident rays R O, R′ O, R″ O, in the more refractive medium, will emerge in the directions O S, O S′, O S″, and the direction of emergence for the incident ray L O will be O B, which is coincident with the bounding surface.

The angle L O N is called the critical angle, and is easily computed when the relative index of refraction is given. For let μ denote this index (the incident ray being supposed to be in the less refractive medium), then we are to have

$$\frac{\sin 90^\circ}{\sin x} = \mu, \text{ whence } \sin x = \frac{1}{\mu};$$

that is, *the sine of the critical angle is the reciprocal of the index of refraction.*

When the media are air and water, this angle is about 48° 30′. For air and different kinds of glass its value ranges from 38° to 41°.

If a ray, as I O, is incident in the more refractive medium, at an angle greater than the critical angle, the law of sines becomes nugatory, and experiment shows that such a ray undergoes internal reflection in the direction O I′, the angle of reflection being equal to

the angle of incidence. Reflection occurring in these circumstances is nearly perfect, and has received the name of *total reflection*. *Total reflection occurs when rays are incident in the more refractive medium at an angle greater than the critical angle.*

The phenomenon of total reflection may be observed in several familiar instances. For example, if a glass of water, with a spoon in it (Fig. 694), is held above the level of the eye, the under side of

Fig. 694.—Total Reflection.

the surface of the water is seen to shine like a brilliant mirror, and the lower part of the spoon is seen reflected in it. Beautiful effects of the same kind may be observed in aquariums.

938. Camera Lucida.—The *camera lucida* is an instrument sometimes employed to facilitate the sketching of objects from nature. It acts by total reflection, and may have various forms, of which that proposed by Wollaston, and represented in Figs. 695, 696, is

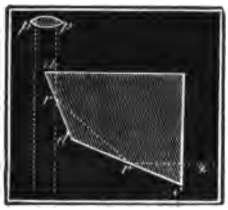


Fig. 695.—Section of Prism.

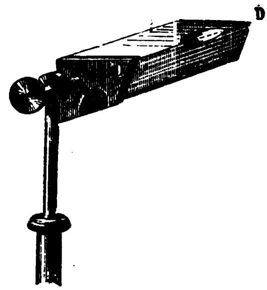


Fig. 696.—Camera Lucida.

one of the commonest. The essential part is a totally-reflecting prism with four angles, one of which is 90°, the opposite one 135°, and the other two each 67° 30′. One of the two faces which contain the right angle is turned towards the objects to be sketched. Rays incident normally on this face, as $x\,r$, make an angle greatly exceeding the critical angle with the face $c\,d$, and are totally reflected from it to the next face $d\,a$, whence they are again totally reflected to the fourth face, from which they emerge normally.[1] An eye placed so as to receive the emergent rays will see a virtual image in a direction at right angles to that in which the object lies. In practice, the eye is held over the angle a of the prism, in such a position that one-half of the pupil receives these reflected rays, while the other half receives light in a parallel direction outside the prism. The observer thus sees the reflected image projected on a real back-ground, which consists of a sheet of paper for sketching. He is thus enabled to pass a pencil over the outlines of the image; pencil, image, and paper being simultaneously visible. It is very desirable that the image should lie in the plane of the paper, not only because the pencil point and the image will then be seen with the same focussing of the eye, but also because parallax is thus obviated, so that when the observer shifts his eye the pencil point is not displaced on the image. A concave lens, with a focal length of something less than a foot, is therefore

[1] The use of having *two* reflections is to obtain an erect image. An image obtained by one reflection would be upside down.

placed close in front of the prism, in drawing distant objects. By raising or lowering the prism in its stand (Fig. 696), the image of the object to be sketched may be made to coincide with the plane of the paper.

The prism is mounted in such a way that it can be rotated either about a horizontal or a vertical axis; and its top is usually covered with a movable plate of blackened metal, having a semicircular notch at one edge, for the observer to look through.

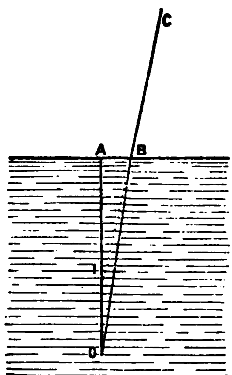


Fig. 697.—Image by Refraction.

989. Images by Refraction at a Plane Surface.—Let O (Fig. 697) be a small object in the interior of a solid or liquid bounded by a plane surface A B. Let O B C be the path of a nearly normal ray, and let B C (the portion in air) be produced backwards to meet the normal in I. Then, since A I B and A O B are the inclinations of the two portions of the ray to the normal, we have (if μ be the index of refraction from air into the substance)—

$$\mu = \frac{\sin AIB}{\sin AOB} = \frac{OB}{IB}.$$

But O B is ultimately equal to O A, and I B to I A. Hence, if we make A I equal to $\frac{AO}{\mu}$, all the emergent rays of a small and nearly normal pencil emitted by O will, if produced backwards, intersect O A at points indefinitely near to the point I thus determined. If the eye of an observer be situated on the production of the normal O A, the rays by which he sees the object O constitute such a pencil. He accordingly sees the image at I. As the value of μ is $\frac{4}{3}$ for water, and about $\frac{3}{2}$ for glass, it follows that the apparent depth of a pool of clear water when viewed vertically is $\frac{3}{4}$ of the true depth, and that the apparent thickness of a piece of plate-glass when viewed normally is only $\frac{2}{3}$ of the true thickness.

990.—When the incident pencil (Fig. 698) is not small, but includes rays of all obliquities, those of them which make angles with the normal less than the critical angle N Q R will emerge into air; and the emergent rays, if produced backwards, will all touch a certain

caustic surface, which has the normal Q N for its axis of revolution, and touches the surface at all points of a circle of which N R is the

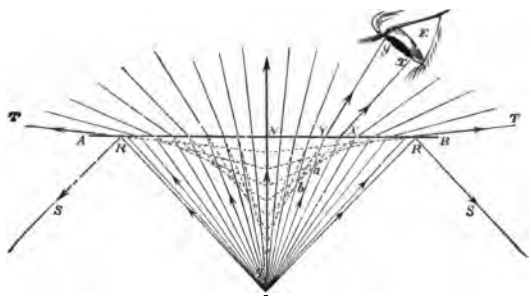


Fig. 698.—Caustic by Refraction.

radius. Wherever the eye may be situated, a tangent drawn from it to the caustic will be the direction of the visible image. If the observer sees the image with both eyes, both being equidistant from the surface and also equidistant from the normal, the two lines of sight thus determined (one for each eye) will meet at a point on the normal, which will accordingly be the apparent position of the image. If, on the other hand, both eyes are in the same plane containing the normal, the two lines of sight will intersect at a point between the normal and the observer.

The image, whether seen with one eye or two, approaches nearer to the surface as the direction of vision becomes more oblique, and ultimately coincides with it. The apparent depth of water, which is only $\frac{3}{4}$ of the real depth when seen vertically, is accordingly less than $\frac{3}{4}$ when seen obliquely, and becomes a vanishing quantity as the direction of vision approaches to parallelism with the surface. The focus I determined in the preceding section is at the cusp of the caustic.

991. Parallel Plate.—Rays falling normally on a uniform transparent plate with parallel faces, keep their course unchanged; but this is not the case with rays incident obliquely. A ray S I (Fig. 699), incident at the angle S I N, is refracted in the direction I R. The angle of incidence at R is equal to the angle of refraction at I, and hence the angle of emergence S′ R N′ is equal to the original angle of incidence S I N. The emergent ray R S′ is therefore parallel to the incident ray S I, but is not in the same straight line with it.

Objects seen obliquely through a plate are therefore displaced from their true positions. Let S (Fig. 700) be a luminous point

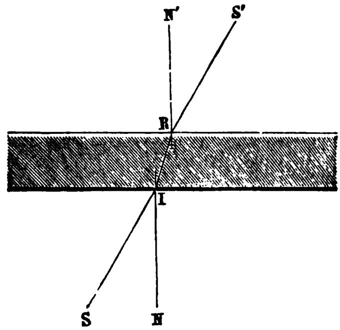


Fig. 699.—Parallel Plate.

Fig. 700.—Vision through Plate.

which sends light to an eye not directly opposite to it, on the other side of a parallel plate. The emergent rays which enter the eye are parallel to the incident rays; but as they have undergone lateral displacement, their point of concourse[1] is changed from S to S′, which is accordingly the image of S.

The displacement thus produced increases with the thickness of the plate, its index of refraction, and the obliquity of incidence. It furnishes one of the simplest means of measuring the index of refraction of a substance, and is thus employed in Pichot's refractometer.

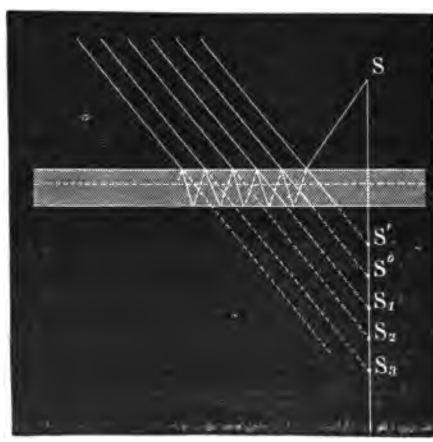


Fig. 701.—Multiple Images in Plate.

992. Multiple Images produced by a Plate.—Let S (Fig. 701) be a luminous point in front of a transparent plate with parallel faces. Of the rays which it sends to the plate, some will be reflected from the front, thus giving rise to an image S′. Another portion will enter the plate,

[1] The rays which compose the pencil that enters the eye will not exactly meet (when produced backwards) in any one point. There will be two focal lines, just as in the case of spherical mirrors (§ 974, 975).

undergo reflection at the back, and emerge with refraction at the front, giving rise to a second image S^0. Another portion will undergo internal reflection at the front, then again at the back, and by emerging in front will form a third image S_1. The same process may be repeated several times; and if the luminous object be a candle, or a piece of bright metal, a number of images, one behind another, will be visible to an eye properly placed in front (Fig. 702). All the successive images, after the first two, continually diminish in brightness. If the glass be silvered at the back, the second image is much brighter than the first, when the incidence is nearly normal, but as the angle of incidence increases, the first image gains upon the second, and ultimately surpasses it. This is due to the fact that the reflecting power of a surface of glass increases with the angle of incidence.

Fig. 702.
Images of Candle in Looking-glass.

If the luminous body is at a distance which may be regarded as infinite,—if it is a star, for example,—all the images should coincide, and form only a single image, occupying a position which does not vary with the position of the observer, provided that the plate is perfectly homogeneous, and its faces perfectly plane and parallel. A severe test is thus furnished of the fulfilment of these conditions.

Plates are sometimes tested, for parallelism and uniformity, by supporting them in a horizontal position on three points, viewing the image of a star in them with a telescope furnished with cross wires, and observing whether the image is displaced on the wires when the plate is shifted into a different position, still resting on the same three points.

993. Superimposed Plates. Astronomical Refraction.—We have stated in § 986 that the relative index from one medium into another is equal to the absolute index of the second divided by that of the first. Hence if μ_1 μ_2 are the absolute indices, and ϕ_1 ϕ_2 the angles which the two parts of the refracted ray make with the normal, we have

$$\sin \phi_1 = \frac{\mu_2}{\mu_1} \sin \phi_2,$$

or

$$\mu_1 \sin \phi_1 = \mu_2 \sin \phi_2. \qquad (1)$$

When a number of plates are superimposed, they will have a common normal. Let a ray pass through them all; let μ denote the absolute index of any one of the plates, and φ the angle which the portion of the ray that lies in this plate makes with the normal; then equation (1) shows that $\mu \sin \varphi$ will have the same value for all parts of the ray. Hence if the value of φ for the first plate be given, its value for any plate in the series depends only on the value of μ for that plate, and will not be altered by removing some or all of the intervening plates.

This reasoning can be applied to the transmission of a ray from a star through the earth's atmosphere, if the distance of the star from the zenith does not exceed 20° or 30°. The portion of atmosphere traversed may be regarded as a series of horizontal plates, and the slope of the ray in the lowest plate will be the same as if all the plates above it were removed. In the case of stars near the horizon, the length of the path in air is so great that the curvature of the earth cannot be left out of account, in other words, the layers traversed cannot be regarded as parallel plates.

994. **Refraction through a Prism.**—For optical purposes, any portion of a transparent body lying between two plane faces which are not parallel may be regarded as a prism.[1] The line in which these faces meet, or would meet if produced, is called the edge of the prism, and a section made by a plane perpendicular to them both is called a *principal section.* The prisms chiefly employed are really prisms in the geometrical sense of the word. Their principal sections are usually triangular, and are very frequently equilateral, as in Fig. 703. The stand usually employed for prisms when mounted separately is represented in Fig. 704. It contains several joints. The uppermost is for rotating the prism about its own axis. The second is for turning the prism so that its edges shall make any required angle with the vertical. The third gives motion about a vertical axis, and also furnishes the means of raising and lowering the prism through a range of several inches.

Let S I (Fig. 705) be an incident ray in the plane of a principal section of the prism. If the external medium be air, or any other

[1] This amounts to saying that the word *prism* in optics means *wedge.*

substance of less refractive power than the prism, the ray in entering the prism will be bent nearer to the normal, taking such a course as I E, and in leaving the prism will be bent away from the normal,

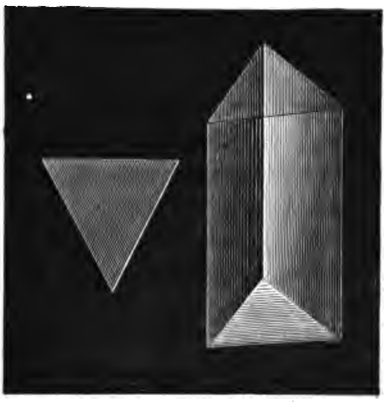


Fig. 703.—Equilateral Prism.

Fig. 704.—Prism mounted on Stand.

taking the course E B. The effect of these two refractions is, therefore, to turn the ray away from the edge (or refracting angle) of the prism. In practice, the prism is usually so placed that I E, the path of the ray through the prism, makes equal angles with the two faces at which refraction occurs (§ 995). If the prism is turned very far from this position, the course of the ray may be altogether different from that represented in the figure; it may, for example, enter at one face, be internally reflected at another, and come out at the third; but we at present exclude such cases from consideration.

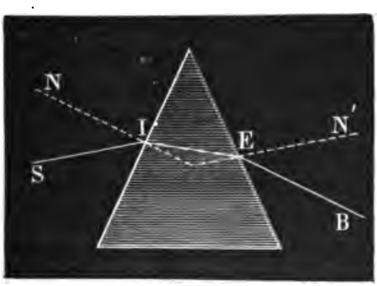


Fig. 705.—Refraction through Prism.

The direction of deviation is easily shown experimentally, by admitting a narrow beam of sunlight into a dark room, and introducing a prism in its course. It will be found that the refracted

beam, in the circumstances represented in Fig. 705, is turned aside some 40° or 50° from its original course.[1]

Since the rays which traverse a prism are bent away from the edge, the object from which they proceed will appear, to an observer looking through the prism, to be more nearly in the direction of the

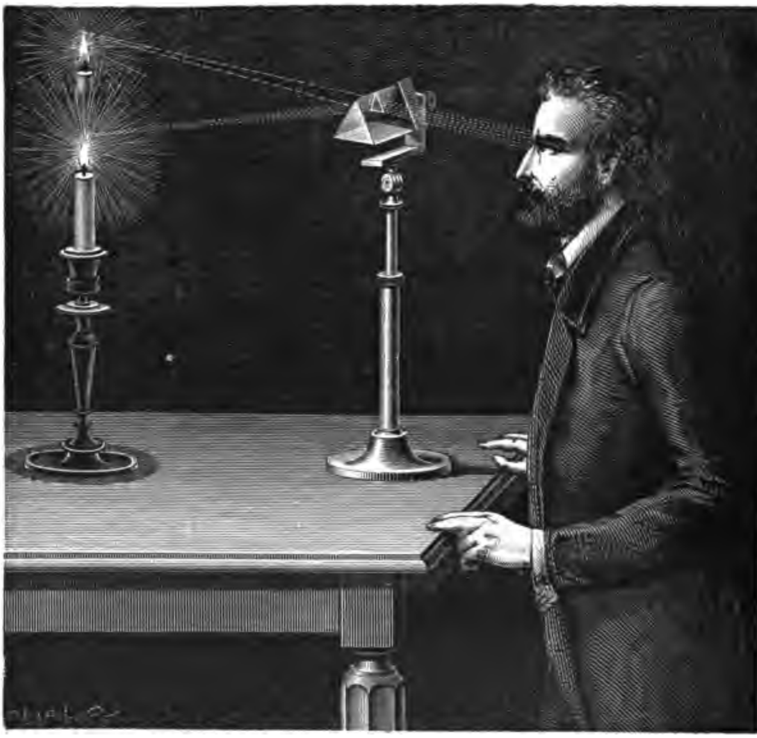


Fig. 706.—Vision through Prism.

edge than it really is. If, for example, he looks at the flame of a candle through a prism placed so that the edge which corresponds to the refracting angle is at the top (Fig. 706), the apparent place of the flame will be above its true place.

995. Formulæ for Refraction through Prisms. Minimum Deviation.—Let S I (Fig. 707) be an incident ray in the plane of a principal section A B C of a prism. Let i be the angle of incidence S I N, and

[1] The phenomena here described are complicated in practice by the unequal refrangibility of rays of different colours (Chap. lxxii.). The complication may be avoided by employing homogeneous light, of which a spirit-lamp, with common salt sprinkled on the wick, affords a nearly perfect example.

r the angle of refraction M I I′. Then, denoting the index of refraction by μ, we have $\sin i = \mu \sin r$. In like manner, putting r' for the angle of internal incidence on the second face I I′ M, and i' for the angle of external refraction N′ I′ R, we have $\sin i' = \mu \sin r'$.

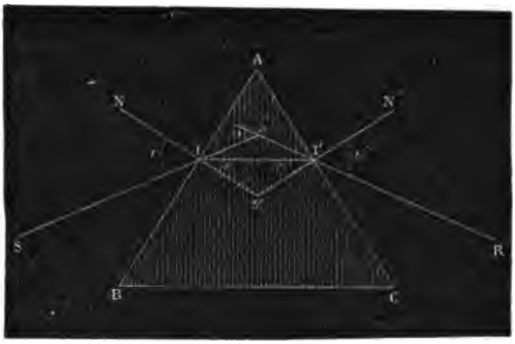


Fig. 707.—Refraction through Prism.

The deviation produced at I is $i - r$, and that at I′ is $i' - r'$, so that the total deviation, which is the acute angle D contained between the rays S I, R I′, when produced to meet at o, is

$$D = i - r + i' - r'. \tag{1}$$

But if we drop a perpendicular from the angular point A on the ray I I′, it will divide the refracting angle B A C into two parts, of which that on the left will be equal to r, and that on the right to r', since the angle contained between two lines is equal to that contained between their perpendiculars. We have therefore $A = r + r'$, and by substitution in the above equation

$$D = i + i' - A. \tag{2}$$

When the path of the ray through the prism I I′ makes equal angles with the two faces, the whole course of the ray is symmetrical with respect to a plane bisecting the refracting angle, so that we have

$$i = i'; \; r = r' = \frac{A}{2}.$$

Equation (2) thus becomes

$$D = 2\,i - A, \text{ whence } i = \frac{A+D}{2}, \tag{3}$$

$$\text{and } \mu = \frac{\sin i}{\sin r} = \frac{\sin \frac{A+D}{2}}{\sin \frac{A}{2}}. \tag{4}$$

This last result is of great practical importance, as it enables us to

calculate the index of refraction μ from measurements of the refracting angle A of the prism, and of the deviation D which occurs when the ray passes symmetrically.

When a beam of sunlight in a dark room is transmitted through a prism, it will be found, on rotating the prism about its axis, that there is a certain mean position which gives smaller deviation of the transmitted light than positions on either side of it; and that, when the prism is in this position, a small rotation of it has no sensible effect on the amount of deviation. The position determined experimentally by these conditions, and known as the *position of minimum deviation*, is the position in which the ray passes symmetrically.

996. Construction for Deviation.—The following geometrical construction furnishes a very simple method of representing the variation of deviation with the angle of incidence:—

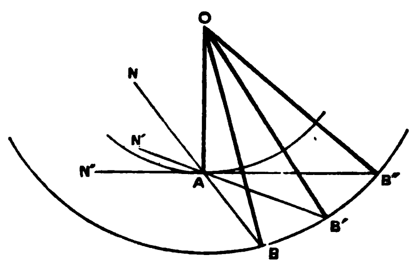


Fig. 708.
General Construction for Deviation.

1. When the refraction is at a single surface, describe two circular arcs about a common centre O (Fig. 708), the ratio of their radii being the index of refraction. Then if the incidence is from rare to dense, draw a radius O A of the smaller circle to represent the direction of the incident ray, and let N A B be the direction of the normal to the surface at the point of incidence, so that O A N is the angle of incidence. Join O B. Then O B N is the angle of refraction, since $\frac{\sin OAN}{\sin OBN} = \frac{OB}{OA}$ = index of refraction; hence O B is parallel to the refracted ray. If the incidence is from dense to rare, we must draw O B to represent the incident ray, make O B N equal to the angle of incidence, and join O A. In either case the angle A O B is the deviation, and it evidently increases with the angle of incidence O A N, attaining its greatest value when this angle (O A N″ in the figure) is a right angle, in which case the angle of refraction O B″ N″ is the critical angle.

2. To find the deviation in refraction through a prism, describe two concentric circular arcs as before (Fig. 709), the ratio of their radii being the index of refraction. Draw the radius O A of the smaller circle to represent the incident ray, N B to represent the

normal at the first surface, B N′ the normal at the second surface. Then O B represents the direction of the ray in the prism, O A′ the direction of the emergent ray, and A O A′ is accordingly the total deviation.

In fact we have

O A N	=	angle of incidence at first surface.
O B N	=	„ refraction „
O B N′	=	„ incidence at second surface.
O A′ N′	=	„ refraction „
A O B	=	deviation at first surface.
B O A′	=	„ second „
A B A′	=	angle between normals = angle of prism.

Again, the deviation A O A′, being the angle at the centre of a circle, is measured by the arc A A′, which subtends it. To obtain the minimum deviation, we must so arrange matters that the angle A B A′ being given (=angle of prism), the arc A A′ shall be a minimum. Let A B A′, *a* B *a*′ (Fig. 710), be two consecutive positions,

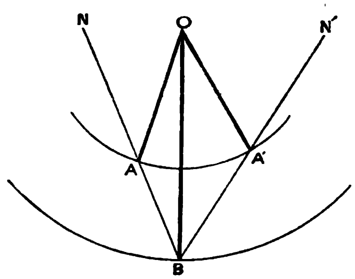


Fig. 709.—Application to Prism.

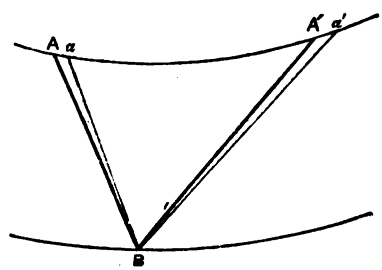


Fig. 710.—Proof of Minimum Deviation.

B A′ and B *a*′ being greater than B A and B *a*. Then, since the small angles A B *a*, A′ B *a*′ are equal, it is obvious, for a double reason, that the small arc A′ *a*′ is greater than A *a*, and hence the whole arc *a a*′ is greater than A A′. The deviation is therefore increased by altering the position in such a way as to make B A and B A′ depart further from equality, and is a minimum when they are equal.

997. Conjugate Foci for Minimum Deviation.—When the angle of incidence is nearly that corresponding to minimum deviation, a small change in this angle has no sensible effect on the amount of deviation.

Hence a small pencil of rays sent in this direction from a luminous point, and incident near the refracting edge, will emerge with their divergence sensibly unaltered, so that if produced backwards they

would meet in a virtual focus at the same distance (but of course not in the same direction) as the point from which they came.

In like manner, if a small pencil of rays converging towards a point, are turned aside by interposing the edge of a prism in the position of minimum deviation, they will on emergence converge to another point at the same distance. We may therefore assert that, neglecting the thickness of a prism, *conjugate foci are at the same distance from it, and on the same side, when the deviation is a minimum.*

998. Double Refraction.—Thus far we have been treating of what is called *single refraction.* We have assumed that to each given incident ray there corresponds only one refracted ray. This is true when the refraction is into a liquid, or into well-annealed glass, or into a crystal belonging to the cubic system.

On the other hand, when an incident ray is refracted into a crystal of any other than the cubic system, or into glass which is unequally stretched or compressed in different directions; for example, into unannealed glass, it gives rise in general to two refracted rays which take different paths; and this phenomenon is called *double refraction.* Attention was first called to it in 1670 by Bartholin, who observed it in the case of Iceland-spar, and its laws for this substance were accurately determined by Huygens.

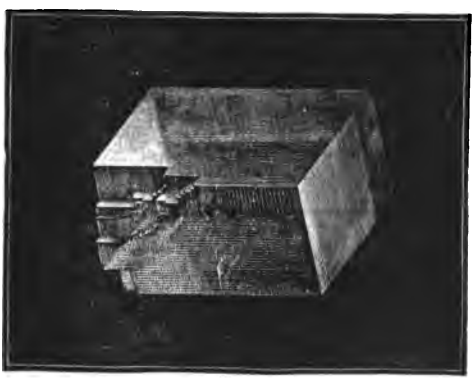


Fig. 711.—Iceland-spar.

999. Phenomena of Double Refraction in Iceland-spar.— Iceland-spar or calc-spar is a form of crystallized carbonate of lime, and is found in large quantity in the country from which it derives its name. It is usually found in rhombohedral form, as represented in Figs. 711, 712.

To observe the phenomenon of double refraction, a piece of the spar may be laid on a page of a printed book. All the letters seen through it will appear double, as in Fig. 712; and the depth of their blackness is considerably less than that of the originals, except where the two images overlap.

In order to state the laws of the phenomena with precision, it is necessary to attend to the crystalline form of Iceland-spar.

At the corner which is represented as next us in Fig. 711 three equal obtuse angles meet; and this is also the case at the opposite

Fig. 712.—Double Refraction of Iceland-spar.

corner which is out of sight. If a line be drawn through one of these corners, making equal angles with the three edges which meet there, it or any line parallel to it is called the axis of the crystal; the axis being properly speaking not a definite *line* but a definite *direction*.

The angles of the crystal are the same in all specimens; but the

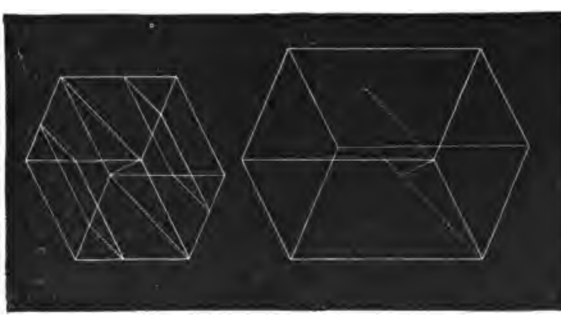


Fig. 713.—Axis of the Crystal.

lengths of the three edges (which may be called the oblique length, breadth, and thickness) may have any ratios whatever. If the crystal is of such proportions that these three edges are equal, as in the first part of Fig. 713, the axis is the direction of one of its diagonals, which is represented in the figure.

Any plane containing (or parallel to) the axis is called a *principal plane* of the crystal.

If the crystal is laid over a dot on a sheet of paper, and is made

to rotate while remaining always in contact with the paper, it will be observed that, of the two images of the dot, one remains unmoved, and the other revolves round it. The former is called the *ordinary*, and the latter the *extraordinary* image. It will also be observed that the former appears nearer than the latter, being more lifted up by refraction.

The rays which form the ordinary image follow the ordinary law of sines (§ 983). They are called the ordinary rays. Those which form the extraordinary image (called the extraordinary rays) do not follow the law of sines, except when the refracting surface is parallel to the axis, and the plane of incidence perpendicular to the axis; and in this case their index of refraction (called the extraordinary index) is different from that of the ordinary rays. The ordinary index is 1·65, and the extraordinary 1·48.

When the plane of incidence is parallel to the axis, the extraordinary ray always lies in this plane, whatever be the direction of the refracting surface; but the ratio of the sines of the angles of incidence and refraction is variable.

When the plane of incidence is oblique to the axis, the extraordinary ray generally lies in a different plane.

We shall recur to the subject of double refraction in the concluding chapter of this volume.

CHAPTER LXX.

LENSES.

1000. Forms of Lenses.—A lens is usually a piece of glass bounded by two surfaces which are portions of spheres. There are two principal classes of lenses.

1. *Converging* lenses or *convex* lenses, which have one or other of the three forms represented in Fig. 714. The first of these is called double convex, the second plano-convex, and the third concavo-convex. This last is also called a converging meniscus. All three

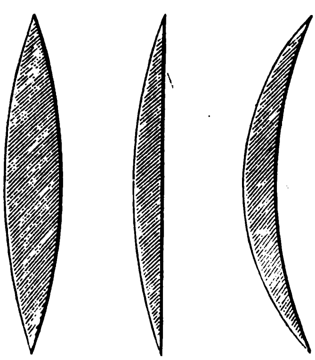


Fig. 714.—Converging Lenses.

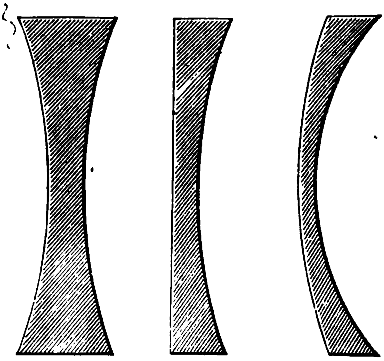


Fig. 715.—Diverging Lenses.

are thicker in the middle than at the edges. They are called converging, because rays are always more convergent or less divergent after passing through them than before.

2. *Diverging* lenses or *concave* lenses (Fig. 715) produce the opposite effect, and are characterized by being thinner in the middle than at the edges. Of the three forms represented, the first is double concave, the second plano-concave, and the third convexo-concave (also called a diverging meniscus).

From the immense importance of lenses, especially convex lenses, in practical optics, it will be necessary to explain their properties at some length.

1001. Principal Focus.—A lens is usually a solid of revolution, and the axis of revolution is called the *axis* of the lens, or sometimes the *principal axis.* When the surfaces are spherical, it is the line joining their centres of curvature.

When rays which were originally parallel to the principal axis pass through a convex lens (Fig. 716), the effect of the two refractions which they undergo, one on entering and the other on leaving the lens, is to make them all converge approximately to one point F, which is called the *principal focus.* The distance A F of the principal focus from the lens is called the *principal focal distance,* or more briefly and usually, the *focal length* of the lens. There is another principal focus at the same distance on the other side of the lens, corresponding to an incident beam coming in the opposite direction. The focal length depends on the convexity of the surfaces of the lens, and also on the refractive power of the material of which it is composed, being shortened either by an increase of refractive power or by a diminution of the radii of curvature of the faces.

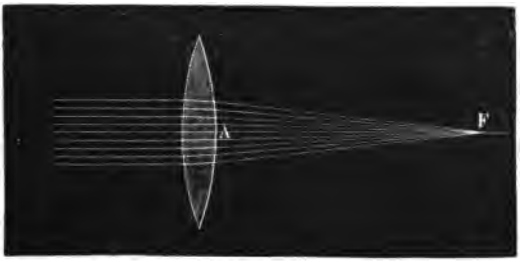


Fig. 716.—Principal Focus of Convex Lens.

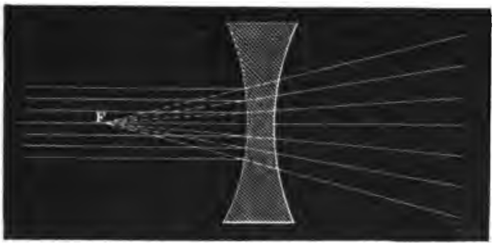


Fig. 717.—Principal Focus of Concave Lens.

In the case of a concave lens, rays incident parallel to the principal axis diverge after passing through; and their directions, if produced backwards, would approximately meet in a point F (Fig. 717), which is still called the principal focus. It is only a virtual focus, inasmuch as the emergent rays do not actually pass through it, whereas the principal focus of a converging lens is real.

1002. Optical Centre of a Lens. Secondary Axes.—Let O and O′ (Fig. 718) be the centres of the two spherical surfaces of a lens. Draw any two parallel radii O I, O′ E to meet these surfaces, and let the joining line I E represent a ray passing through the lens. This ray makes equal angles with the normals at I and E, since these latter are parallel by construction; hence the incident and emergent rays S I, E R also make equal angles with the normals, and are therefore parallel. In fact, if tangent planes (indicated by the dotted lines in the figure) are drawn at I and E, the whole course of the ray SIER will be the same as if it had passed through a plate bounded by these planes.

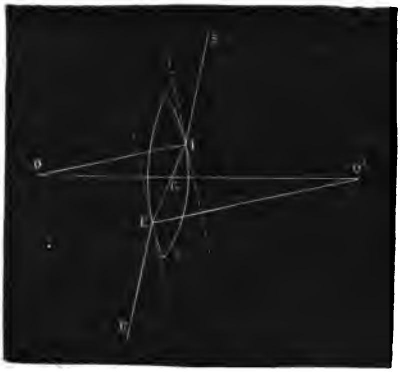


Fig. 718.—Centre of Lens.

Let C be the point in which the line I E cuts the principal axis, and let R, R′ denote the radii of the two spherical surfaces. Then, from the similarity of the triangles O C I, O′ C E, we have

$$\frac{OC}{CO'} = \frac{R}{R'}; \qquad (1)$$

which shows that the point C divides the line of centres O O′ in a definite ratio depending only on the radii. Every ray whose direction on emergence is parallel to its direction before entering the lens, must pass through the point C in traversing the lens; and conversely, every ray which, in its course through the lens, traverses the point C, has parallel directions at incidence and emergence. The point C which possesses this remarkable property is called the *centre*, or *optical centre*, of the lens.

In the case of a double convex or double concave lens, the optical centre lies in the interior, its distances from the two surfaces being directly as their radii. In plano-convex and plano-concave lenses it is situated on the convex or concave surface. In a meniscus of either kind it lies outside the lens altogether, its distances from the surfaces being still in the direct ratio of their radii of curvature.[1]

[1] These consequences follow at once from equation (1); for the distances of C from the

In elementary optics it is usual to neglect the thickness of the lens. The incident and emergent rays S I, E R may then be regarded as lying in one straight line which passes through C, and we may lay down the proposition that *rays which pass through the centre of a lens undergo no deviation.* Any straight line through the centre of a lens is called a *secondary axis.*

The approximate convergence of the refracted rays to a point, when the incident rays are parallel, is true for all directions of in-

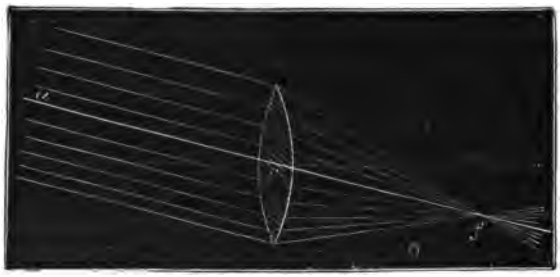


Fig. 719.—Principal Focus on Secondary Axis.

cidence; and the point to which the emergent rays approximately converge (f, Fig. 719) is always situated on the secondary axis (acf) parallel to the incident rays. The focal distance is sensibly the same as for rays parallel to the principal axis, unless the obliquity is considerable.

1003. Conjugate Foci.—When a luminous point S sends rays to a

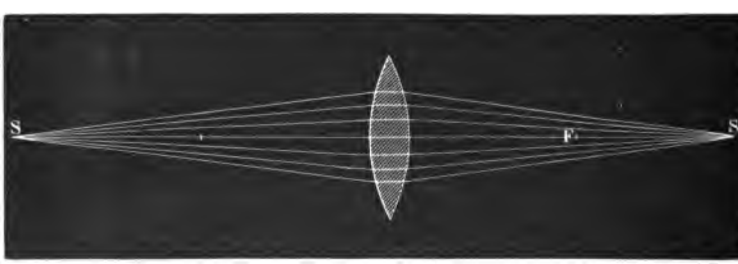


Fig. 720.—Conjugate Foci, both Real.

lens (Fig. 720), the emergent rays converge (approximately) to one

two faces are respectively the difference between R and O C, and the difference between R′ and O′ C, and we have

$$\frac{R}{R'} = \frac{O\,C}{O'C} = \frac{R - O\,C}{R' - O'C}.$$

point S′; whence it follows that rays sent from S′ to the lens would converge (approximately) to S. Two points thus related are called *conjugate foci* of the lens, and the line joining them always passes through the centre of the lens; in other words, they must either be both on the principal axis, or both on the same secondary axis.

The fact that rays which come from one point go to one point is the foundation of the theory of images, as we have already explained in connection with mirrors (§ 967).

The diameters of object and image are directly as their distances from the centre of the lens, and the image will be erect or inverted according as the object and image lie on the same side or on opposite sides of this centre (§ 971). There is also, in the case of lenses, the same difference between an image seen in mid-air and an image thrown on a screen which we have pointed out in § 974.

It is to be remarked that the distinction between principal and secondary axes has much more significance in the case of lenses than of mirrors; and images produced by a lens are more distinct in the neighbourhood of the principal axis than at a distance from it.

1004. Formulæ relating to Lenses.—The deviation produced in a ray by transmission through a lens will not be altered by substituting

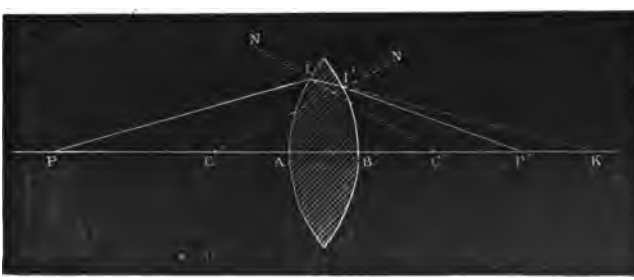


Fig. 721.—Diagram showing Path of Ray, and Normals.

for the lens a prism bounded by planes which touch the lens at the points of incidence and emergence; and in the actual use of lenses, the direction of the rays with respect to the supposed prism is such as to give a deviation not differing much from the minimum. The expression for the minimum deviation (§ 995) is $2\,i - 2\,r$ or $2\,i - A$; and when the angle of the prism is small, as it is in the case of ordinary lenses, we may assume $\frac{i}{r} = \frac{\sin i}{\sin r} = \mu$; so that $2\,i$ becomes $2\,\mu r$ or $\mu\,A$, and the expression for the deviation becomes

$$(\mu - 1)\,A, \qquad (1)$$

A being the angle between the tangent planes (or between the normals) at the points of entrance and emergence.

Let x_1 and x_2 denote the distances of these points respectively from the principal axis, and r_1, r_2 the radii of curvature of the faces on which they lie. Then $\frac{x_1}{r_1}$, $\frac{x_2}{r_2}$ are the sines of the angles which the normals make with the axis, and the angle A is the sum or difference of these two angles, according to the shape of the lens. In the case of a double convex lens it is their sum, and if we identify the sines of these small angles with the angles themselves, we have

$$A = \frac{x_1}{r_1} + \frac{x_2}{r_2}. \qquad (2)$$

But if p_1, p_2 denote the distances from the faces of the lens to the points where the incident and emergent rays cut the principal axis, $\frac{x_1}{p_1}$, $\frac{x_2}{p_2}$ are the sines of the angles which these rays make with the axis, and the deviation is the sum or difference of these two angles, according as the conjugate foci are on opposite sides or on the same side of the lens. In the former case, identifying the angles with their sines, the deviation is $\frac{x_1}{p_1} + \frac{x_2}{p_2}$, and this, by formula (1), is to be equal to $(\mu - 1)$ A, that is, to $(\mu - 1)\left(\frac{x_1}{r_1} + \frac{x_2}{r_2}\right)$.

If the thickness of the lens is negligible in comparison with p_1, p_2, we may regard x_1 and x_2 as equal, and the equation

$$\frac{x_1}{p_1} + \frac{x_2}{p_2} = (\mu - 1)\left(\frac{x_1}{r_1} + \frac{x_2}{r_2}\right) \qquad (3)$$

will reduce to

$$\frac{1}{p_1} + \frac{1}{p_2} = (\mu - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right). \qquad (4)$$

If p_1 is infinite, the incident rays are parallel, and p_2 is the principal focal length, which we shall denote by f. We have therefore

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \qquad (5)$$

and

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{f}. \qquad (6)$$

1005. Conjugate Foci on Secondary Axis.—Let M (Fig. 722) be a luminous point on the secondary axis M O M′, O being the centre of

the lens, and let M′ be the point in which an emergent ray corresponding to the incident ray M I cuts this axis. Let x denote x_1 or x_2, the distances of the points of incidence and emergence from the

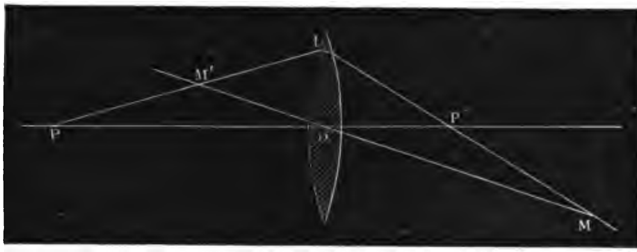


Fig. 722.—Conjugate Foci on Secondary Axis.

principal axis, and θ the obliquity of the secondary axis; then $x \cos \theta$ is the length of the perpendicular from I upon M M′, and $\frac{x \cos \theta}{\text{M I}}, \frac{x \cos \theta}{\text{M}'\text{I}}$, are the sines of the angles O M I, O M′ I respectively. But the deviation is the sum of these angles; hence, proceeding as in last section, we have

$$\frac{x \cos \theta}{\text{M I}} + \frac{x \cos \theta}{\text{M}'\text{I}} = (\mu - 1)\left(\frac{x}{r_1} + \frac{x}{r_2}\right) = \frac{x}{f} \qquad (7)$$

$$\frac{1}{\text{M I}} + \frac{1}{\text{M}'\text{I}} = \frac{1}{f \cos \theta}. \qquad (8)$$

The fact that x does not appear in equations (6) and (8) shows that, for every position of a luminous point, there is a conjugate focus, lying on the same axis as the luminous point itself. Equation (8) shows that the effective focal length becomes shorter as the obliquity becomes greater, its value being $f \cos \theta$, where θ is the obliquity.

If we take account of the fact that the rays of an oblique pencil make the angles of incidence and emergence more unequal than the rays of a direct pencil and thus (by the laws of prisms) undergo larger deviation, we obtain a still further shortening of the effective focal length for oblique pencils.

When the obliquity is small, $\cos \theta$ may be regarded as unity, and we may employ the formula

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{f} \qquad (6)$$

for oblique as well as for direct pencils.

1006. Discussion of the Formula for Convex Lenses.—For convex

lenses f is to be regarded as positive; p will be positive when measured from the lens towards the incident light, and p' when measured in the direction of the emergent light.

Formula (6), being identical with equation (b) of § 968, leads to results analogous to those already deduced for concave mirrors.

As one focus advances from infinite distance to a principal focus, its conjugate moves away from the other principal focus to infinite distance on the other side. The more distant focus is always moving more rapidly than the nearer, and the least distance between them is accordingly attained when they are equidistant from the lens; in which case the distance of each of them from the lens is $2f$, and their distance from each other $4f$.

If either of the distances, as p, is less than f, the formula shows that the other distance p' is negative. The meaning is that the two foci are on the same side of the lens, and in this case one of them (the more distant of the two) must be virtual. For example, in Fig. 723, if S, S′ are a pair of conjugate foci, one of them S being between the principal focus F and the lens, rays sent to the lens by a luminous point at S, will, after emergence, diverge as if from S′; and rays coming from the other side of the lens, if they converge to S′ before incidence, will in reality be made to meet in S. As S moves towards the lens, S′ moves in the same direction more rapidly; and they become coincident at the surface of the lens. The formula in fact shows that if $\frac{1}{p}$ is very great in comparison with $\frac{1}{f}$, and positive, $\frac{1}{p'}$ must be very great and negative; that is to say, if p is a very small positive quantity, p' is a very small negative quantity.

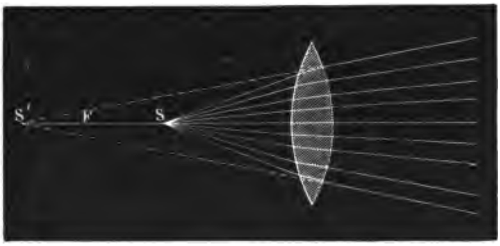


Fig. 723.—Conjugate Foci, one Real, one Virtual.

1007. Formation of Real Images.—Let A B (Fig. 724) be an object in front of a lens, at a distance exceeding the principal focal length. It will have a real image on the other side of the lens. To determine the position of the image by construction, draw through any point A of the object a line parallel to the principal axis, meeting

the lens in A′. The ray represented by this line will after refraction, pass through the principal focus F; and its intersection with the secondary axis A O determines the position of a, the focus conjugate to A. We can in like manner determine the position of b, the focus conjugate to B, another point of the object; and the joining line $a\,b$ will then be the image of the line A B. It is evident that if $a\,b$ were the object, A B would be the image.

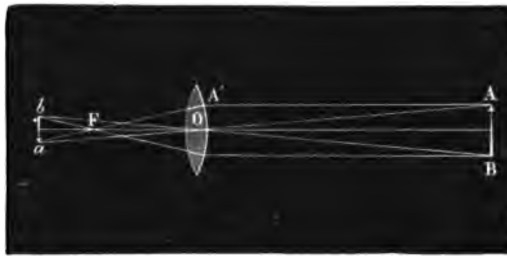


Fig. 724.—Real and Diminished Image.

Figs. 724, 725 represent the cases in which the distance of the object is respectively greater and less than twice the focal length of the lens.

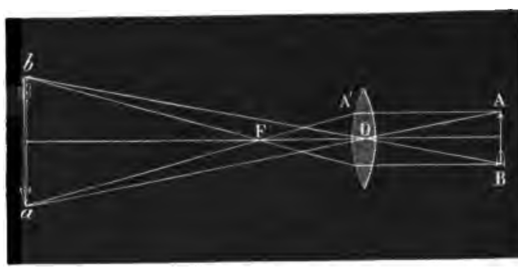


Fig. 725.—Real and Magnified Image.

1008. Size of Image.—In each case it is evident that $\frac{A\,B}{a\,b}=\frac{O\,A}{O\,a}=\frac{p}{p'}$, or *the linear dimensions of object and image are directly as their distances from the centre of the lens.*

Again, since by equation (6)

$$\frac{1}{p'}=\frac{1}{f}-\frac{1}{p}=\frac{p-f}{pf} \qquad (9)$$

we have

$$\frac{p}{p'}=\frac{p-f}{f}$$

and

$$a\,b=\frac{f}{p-f}A\,B; \qquad (10)$$

from which formula the size of the image can be calculated without finding its position.

1009. Example.—A straight line 25$^{mm.}$ long is placed perpendi-

cularly on the axis, at a distance of 35 centimetres from a lens of 15 centimetres' focal length; what are the position and magnitude of the image?

To determine the distance p' we have

$$\frac{1}{35}+\frac{1}{p'}=\frac{1}{15};\text{ whence } p'=\frac{35\times 15}{35-15}=26\tfrac{1}{4}^{\text{cm.}}$$

For the length of the image we have

$$25\frac{f}{p-f}=25\frac{15}{35-15}=18\tfrac{3}{4}^{\text{mm.}}$$

1010. Image on Cross-wires.—The position of a real image seen in mid-air can be tested by means of a cross of threads, or other convenient mark, so arranged that it can be fixed at any required point. The observer must fix this cross so that it appears approximately to coincide with a selected point of the image. He must then try whether any relative displacement of the two occurs on shifting his eye to one side. If so, the cross must be pushed nearer to the lens, or drawn back, according to the nature of the observed displacement, which follows the general rule of parallactic displacement, that the more distant object is displaced in the same direction as the observer's eye. The cross may thus be brought into exact coincidence with the selected point of the image, so as to remain in apparent coincidence with it from all possible points of view. When this coincidence has been attained, the cross is at the focus conjugate to that which is occupied by the selected point of the object.

By employing two crosses of threads, one to serve as object, and the other to mark the position of the image, it is easy to verify the fact that when the second cross coincides with the image of the first, the first also coincides with the image of the second.

1011. Aberration of Lenses.—In the investigations of §§ 1004, 1005, we made several assumptions which were only approximately true. The rays which proceed from a luminous point to a lens are in fact not accurately refracted to one point, but touch a curved surface called a caustic. The cusp of this caustic is the conjugate focus, and is the point at which the greatest concentration of light occurs. It is accordingly the place where a screen must be set to obtain the brightest and most distinct image. Rays from the central parts of the lens pass very nearly through it; but rays from the circumferen-

tial portions fall short of it. This departure from exact concurrence is called *aberration.* The distinctness of an image on a screen is improved by employing an annular diaphragm to cut off all except the central rays; but the brightness is of course diminished.

By holding a convex lens in a position very oblique to the incident light, a primary and secondary focal line can be exhibited on a screen perpendicular to the beam, just as in the case of concave mirrors (§ 975). The experiment, however, is rather more difficult of performance.

1012. Virtual Images.—Let an object A B be placed between a convex lens and its principal focus. Then the foci conjugate to the points A, B are virtual, and their positions can be found by construction from the consideration that rays through A, B, parallel to the principal axis, will be refracted to F, the principal focus on the other side. These refracted rays, if produced backward, must meet the secondary axes O A, O B in the required points. An eye placed on the other side of the lens will accordingly see a virtual image, erect, magnified, and at a greater distance from the lens than the object. This is the principle of the simple microscope. The formula for the distances D, d of object and image from the lens, when both are on the same side, is

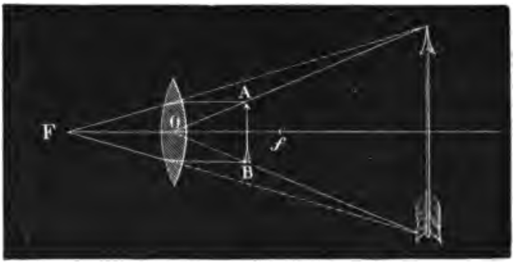


Fig. 726.—Virtual Image formed by Convex Lens.

$$\frac{1}{D} - \frac{1}{d} = \frac{1}{f}, \qquad (11)$$

f denoting the principal focal length.

1013. Concave Lens.—For a concave lens, if the focal length be still regarded as positive, and denoted by f, and if the distances D, d be on the same side of the lens, the formula becomes

$$\frac{1}{d} - \frac{1}{D} = \frac{1}{f}, \qquad (12)$$

which shows that d is always less than D; that is, the image is nearer to the lens than the object.

In Fig. 727, A B is the object, and *a b* the image. Rays incident from A and B parallel to the principal axis will emerge as if they came from the principal focus F. Hence the points *a b* are determined by the intersections of the dotted lines in the figure with the secondary axes O A, O B. An eye on the other side of the lens sees the image *a b*, which is always virtual, erect and diminished.

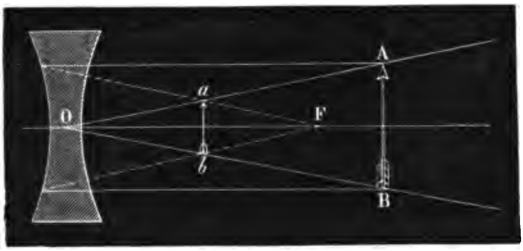


Fig. 727.—Virtual Image formed by Concave Lens.

1014. Focometer.—Silbermann's focometer (Fig. 728) is an instrument for measuring the focal lengths of convex lenses, and is based on the principle (§ 1006) that, when the object and its image are equidistant from the lens, their distance from each other is four times the focal length. It consists of a graduated rule carrying three runners M, L, M′. The middle one L is the support for the lens which is to be examined; the other two, M M′, contain two thin plates of horn or other translucent material, ruled with lines, which are at the same distance apart in both. The sliders must be adjusted until the image of one of these plates is thrown upon the other plate, without enlargement or

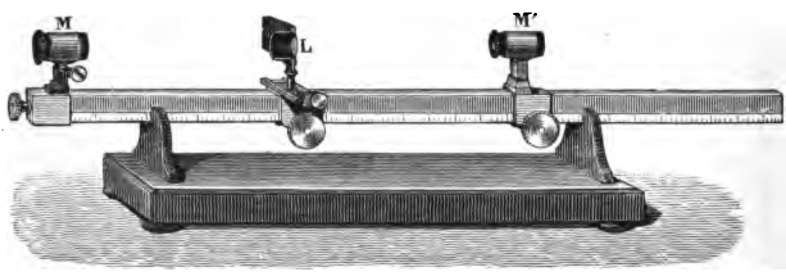


Fig. 728.—Silbermann's Focometer.

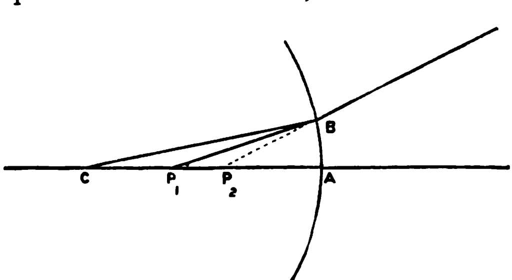


Fig. 729.

diminution, as tested by the coincidence of the ruled lines of the image with those of the plate on which it is cast. The distance between M and M′ is then read off, and divided by 4.

1015. Refraction at a Single Spherical Surface.—Suppose a small pencil of rays to be incident nearly normally upon a spherical surface which forms the boundary between two media in which the indices are μ_1 and μ_2 respectively. Let C (Fig. 729) be the centre of curvature, and C A the axis. Let P_1 be the focus of the incident, and P_2 of the refracted rays. Then for any ray P_1B, CBP_1 is the angle of incidence and CBP_2 the angle of refraction. Hence by the law of sines we have (§ 993)

$$\mu_1 \sin CBP_1 = \mu_2 \sin CBP_2.$$

Dividing by sin B C A, and observing that

$$\frac{\sin CBP_1}{\sin BCA} = \frac{CP_1}{BP_1} = \frac{CP_1}{AP_1} \text{ ultimately;}$$

$$\frac{\sin CBP_2}{\sin BCA} = \frac{CP_2}{BP_2} = \frac{CP_2}{AP_2} \text{ ultimately;}$$

we obtain the equation

$$\mu_1 \frac{CP_1}{AP_1} = \mu_2 \frac{CP_2}{AP_2}, \tag{13}$$

which expresses the fundamental relation between the positions of the conjugate foci.

Let $AC = r$, $AP_1 = p_1$, $AP_2 = p_2$, then equation (13) becomes

$$\mu_1 \frac{r - p_1}{p_1} = \mu_2 \frac{r - p_2}{p_2}, \tag{14}$$

or, dividing by r,

$$\mu_1 \left(\frac{1}{p_1} - \frac{1}{r}\right) = \mu_2 \left(\frac{1}{p_2} - \frac{1}{r}\right),$$

which may be written

$$\frac{\mu_2}{p_2} - \frac{\mu_1}{p_1} = \frac{\mu_2 - \mu_1}{r}. \tag{15}$$

Again, let $CA = \rho$, $CP_1 = q_1$, $CP_2 = q_2$, then equation (13) gives

$$\mu_1 \frac{q_1}{\rho - q_1} = \mu_2 \frac{q_2}{\rho - q_2},$$

or

$$\frac{1}{\mu_1} \frac{\rho - q_1}{q_1} = \frac{1}{\mu_2} \frac{\rho - q_2}{q_2}, \tag{16}$$

an equation closely analogous to (14) and leading to the result (analogous to (15))

$$\frac{1}{\mu_2 q_2} - \frac{1}{\mu_1 q_1} = \left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right) \frac{1}{\rho}. \tag{17}$$

The signs of p_1, p_2, r, in (14) and (15) are to be determined by the

rule that, if one of the three points P_1, P_2, C lies on the opposite side of A from the other two, its distance from A is to be reckoned opposite in sign to theirs.

In like manner the signs of q_1, q_2, ρ, in (16) and (17) are to be determined by the rule that, if one of the three points P_1, P_2, A lies on the opposite side of C from the other two, its distance from C is to be reckoned opposite in sign to theirs.

It is usual to reckon distances positive when measured *towards the incident light;* but the formulæ will remain correct if the opposite convention be adopted.

If f denote the principal focal length, measured from A, we have, by (15), writing f for p_2 and making p_1 infinite,

$$\frac{1}{f} = \frac{\mu_2 - \mu_1}{\mu_2} \frac{1}{r},$$

and (15) may now be written

$$\frac{\mu_2}{p_2} - \frac{\mu_1}{p_1} = \frac{\mu_2}{f},$$

it being understood that the positive direction for f is the same as for p_1, p_2, and r.

The application of these formulæ to lenses in cases where the thickness of the lens cannot be neglected, may be illustrated by the following example.

1016. To find the position of the image formed by a spherical lens.

Let distances be measured from the centre of the sphere, and be reckoned positive on the side next the incident light.

Then, if x denote the distance of the object, y the distance of the image formed by the first refraction, z the distance of the image formed by the second refraction, a the radius of the sphere, and μ its index of refraction; we have, at the first surface,

$$\rho = a \qquad \mu_1 = 1 \qquad \mu_2 = \mu,$$

and at the second surface

$$\rho = -a \qquad \mu_1 = \mu \qquad \mu_2 = 1.$$

Hence equation (17) gives, for the first refraction,

$$\frac{1}{\mu y} - \frac{1}{x} = \left(\frac{1}{\mu} - 1\right)\frac{1}{a},$$

and for the second refraction,

$$\frac{1}{z} - \frac{1}{\mu y} = -\left(1 - \frac{1}{\mu}\right)\frac{1}{a} = \left(\frac{1}{\mu} - 1\right)\frac{1}{a}.$$

By adding these two equations, we obtain

$$\frac{1}{z} - \frac{1}{x} = \left(\frac{1}{\mu} - 1\right)\frac{2}{a} = -\frac{\mu - 1}{\mu} \cdot \frac{2}{a}.$$

If the incident rays are parallel, we have x infinite and $z = -\frac{\mu}{\mu-1}\frac{a}{2}$; that is to say, the principal focus is at a distance $\frac{\mu}{\mu-1}\frac{a}{2}$ from the centre, on the side remote from the incident light.

1017. Camera Obscura.—The images obtained by means of a hole in the shutter of a dark room (§ 938) become sharper as the size of the hole is diminished; but this diminution involves loss of light, so that it is impossible by this method to obtain an image at once bright and sharp. This difficulty can be overcome by employing a lens. If the objects in the external landscape depicted are all at distances many times greater than the focal length of the lens, their images will all be formed at sensibly the same distance from the lens, and may be received upon a screen placed at this distance. The images thus obtained are inverted, and are of the same size as if a simple aperture were employed instead of a lens. This is the principle on which the *camera obscura* is constructed.

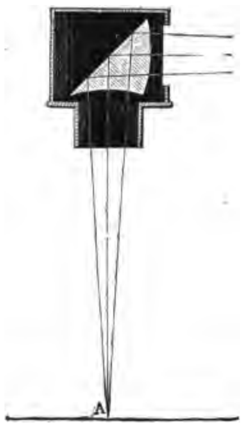


Fig. 730.—Objective of Camera.

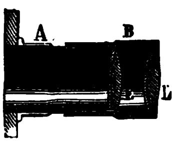


Fig. 731.—Photographic Camera.

It is a kind of tent surrounded by opaque curtains, and having at its top a revolving lantern, containing a lens with its axis horizontal, and a mirror placed behind it at a slope of 45°, to reflect the transmitted light downwards on to a sheet of white paper lying on the top of a table. Images of external objects are thus depicted on the paper, and their outlines can be traced with a pencil if desired. It is still better to combine lens and mirror in one, by the arrangement represented in section in Fig. 730. Rays

from external objects are first refracted at a convex surface, then totally reflected at the back of the lens, which is plane, and finally emerge through the bottom of the lens, which is concave, but with a larger radius of curvature than the first surface. The two refractions produce the effect of a converging mensicus. The instrument is now only employed for purposes of amusement.

1018. Photographic Camera.—The camera obscura employed by photographers (Fig. 731) is a box MN with a tube AB in front, containing an object-glass at its extremity. The object-glass is usually compound, consisting of two single lenses E, L, an arrangement which is very commonly adopted in optical instruments, and which has the advantage of giving the same effective focal length as a single lens of smaller radius of curvature, while it permits the employment of a larger aperture, and consequently gives more light. At G is a slide of ground glass, on which the image of the scene to be depicted is thrown, in setting the instrument. The focussing is performed in the first place by sliding the part M of the box in the part N, and finally by the pinion V which moves the lens. When the image has thus been rendered as sharp as possible, the sensitized plate is substituted for the ground glass.[1]

[1] The photographic processes at present in use are very various, both optically and chemically; but are all the same in principle with the method originally employed by Talbot. This method, which was almost forgotten during the great success of Daguerre, consists in first obtaining, on a transparent plate, a picture with lights and shades reversed, called a *negative;* then placing this upon a piece of paper sensitized with chloride of silver, and exposing it to the sun's rays. The light parts of the negative allow the light to pass and blacken the paper, thus producing a positive picture. The same negative serves for producing a great number of positives.

The negative plate is usually a glass plate covered with a film of collodion (sometimes of albumen), sensitized by a salt of silver. The following is one of the numerous formulæ for this preparation. Take

Sulphuric ether,	300 grammes
Alcohol at 40°,	200 „
Gun cotton,	5 „

Incorporate these ingredients thoroughly in a porcelain mortar; then add

Iodide of potassium,	13 grammes
Iodide of ammonium,	1·75 „
Iodide of cadmium,	1·75 „
Bromide of cadmium,	1·25 „

This mixture is poured over the plate, which is then immersed in a solution (10 per cent

1019. Use of Lenses for Purposes of Projection.—Lenses are extensively employed in the lecture-room, for rendering experiments visible to a whole audience at once, by projecting them on a screen. The arrangements vary according to the circumstances of each case, and cannot be included in a general description.

1020. Solar Microscope. Magic Lantern.—In the solar microscope, a convex lens of short focal length is employed to throw upon a screen a highly-magnified image of a small object placed a little beyond the principal focus. As the image is always much less bright than the

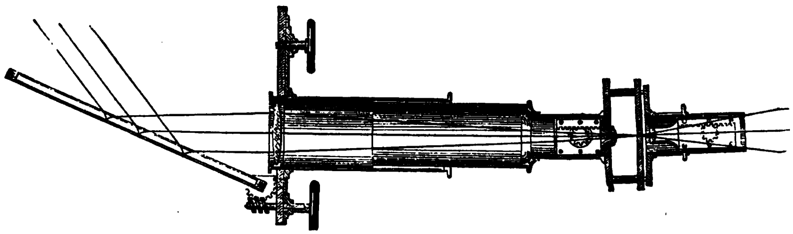


Fig. 732.—Solar Microscope.

object, and the more so as the magnification is greater, it is necessary that the object should be very highly illuminated. For this purpose the rays of the sun are directed upon it by means of a mirror and large lens; the latter serving to increase the solid angle of the cone of rays which fall upon the object, and thus to enable a larger portion of the magnifying lens to be utilized. The objects magnified are always transparent; and the images are formed by rays which have been transmitted through them.

strong) of nitrate of silver. The film of collodion is thus brought to an opal tint, and the plate, after being allowed to drain, is ready for exposure in the camera.

After being *exposed*, the picture is *developed*, by the application of a liquid for which the following is a formula;

Distilled water, ...	250 grammes.
Pyrogallic acid, ...	1 „
Crystallizable acetic acid, ...	20 „

When the picture is sufficiently developed, it is *fixed*, by the application of a solution, either of hyposulphite of soda from 25 to 30 per cent strong, or of cyanide of potassium 3 per cent strong, and the negative is completed.

To obtain a positive, the negative plate is laid upon a sheet of paper in a glass dish, the paper having been sensitized by immersing it first in a solution of common sea-salt 3 or 4 per cent strong, and then in a solution of nitrate of silver 18 per cent strong. The exposure is continued till the tone is sufficiently deep, the tint is then improved by means of a salt of gold, and the picture is fixed by hyposulphite of soda. It has then only to be washed and dried.—*D.*

The lens employed for producing the image is usually compound, consisting of a convex and a concave lens combined.

The electric light can be employed instead of the sun. The apparatus for regulating this light is usually placed within a lantern (Fig. 733), in such a position that the light is at the centre of curva-

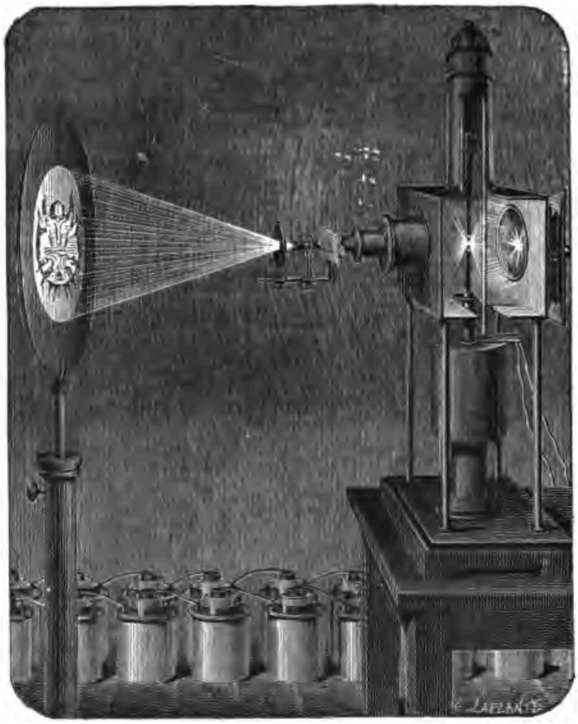


Fig. 733.—Photo-electric Microscope.

ture of a spherical mirror, so that the inverted image of the light coincides with the light itself. The light is concentrated on the object by a system of lenses, and, after passing through the object, traverses another system of lenses, placed at such a distance from the object as to throw a highly-magnified image of it on a screen. The whole arrangement is called the *electric* or *photo-electric microscope.*

The magic lantern is a rougher instrument of the same kind, employed for projecting magnified images of transparent paintings, executed on glass slides. It has one lens for converging a beam of light on the slide, and another for throwing an image of the slide on the screen. In all these cases the image is inverted.

CHAPTER LXXI.

VISION AND OPTICAL INSTRUMENTS.

1021. Description of the Eye.—The human eye (Fig. 734) is a nearly spherical ball, capable of turning in any direction in its socket. Its outermost coat is thick and horny, and is opaque except in its

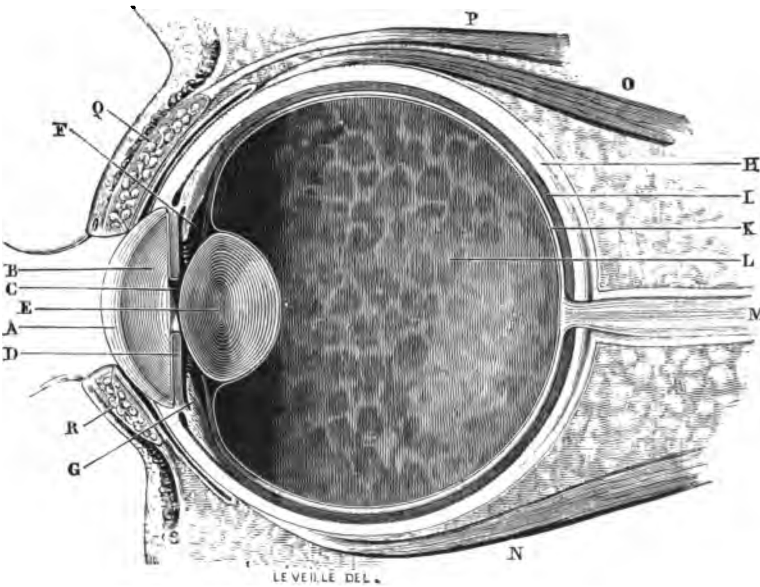


Fig. 734.—Human Eye.

anterior portion. Its opaque portion H is called the *sclerotica,* or in common language the white of the eye. Its transparent portion A is called the *cornea,* and has the shape of a very convex watch-glass. Behind the cornea is a diaphragm D, of annular form, called the *iris.* It is coloured and opaque, and the circular aperture C in its centre

is called the *pupil.* By the action of the involuntary muscles of the iris, this aperture is enlarged or contracted on exposure to darkness or light. The colour of the iris is what is referred to when we speak of the colour of a person's eyes. Behind the pupil is the *crystalline lens* E, which has greater convexity at back than in front. It is built up of layers or shells, increasing in density inwards, the outermost shell having nearly the same index of refraction as the media in contact with it; an arrangement which tends to prevent the loss of light by reflection. The cavity B between the cornea and the crystalline is called the anterior chamber, and is filled with a watery liquid called the *aqueous humour.* The much larger cavity L, behind the crystalline, is called the posterior chamber, and is filled with a transparent jelly called the *vitreous humour,* inclosed in a very thin transparent membrane (the hyaloid membrane). The posterior chamber is inclosed, except in front, by the *choroid coat* or *uvea* I, which is saturated with an intensely black and opaque mucus, called the *pigmentum nigrum.* The choroid is lined, except in its anterior portion, with another membrane K, called the retina, which is traversed by a ramified system of nerve filaments diverging from the optic nerve M. Light incident on the retina gives rise to the sensation of vision; and there is no other part of the eye which possesses this property.

1022. The Eye as an Optical Instrument.—It is clear, from the above description, that a pencil of rays entering the eye from an external point will undergo a series of refractions, first at the anterior surface of the cornea, and afterwards in the successive layers of the crystalline lens, all tending to render them convergent (see table of indices, § 986). A real and inverted image is thus formed of any external object to which the eye is directed. If this image falls on the retina, the object is seen; and if the image thus formed on the retina is sharp and sufficiently luminous, the object is seen distinctly.

1023. Adaptation to Different Distances.—As the distance of an image from a lens varies with the distance of the object, it would only be possible to see objects distinctly at one particular distance, were there not special means of adaptation in the eye. Persons whose sight is not defective can see objects in good definition at all distances exceeding a certain limit. When we wish to examine the minute details of an object to the greatest advantage, we hold it at a particular distance, which varies in different individuals, and averages about eight inches. As we move it further away, we

experience rather more ease in looking at it, though the diminution of its apparent size, as measured by the visual angle, renders its minuter features less visible. On the other hand, when we bring it nearer to the eye than the distance which gives the best view, we cannot see it distinctly without more or less effort and sense of strain; and when we have brought it nearer than a certain lower limit (averaging about six inches), we find distinct vision no longer possible. In looking at very distant objects, if our vision is not defective, we have very little sense of effort. These phenomena are in accordance with the theory of lenses, which shows that when the distance of an object is a large multiple of the focal length of the lens, any further increase, even up to infinity, scarcely alters the distance of the image; but that, when the object is comparatively near, the effect of any change of its distance is considerable. There has been much discussion among physiologists as to the precise nature of the changes by which we adapt our eyes to distinct vision at different distances. Such adaptation might consist either in a change of focal length, or in a change of distance of the retina. Observations in which the eye of the patient is made to serve as a mirror, giving images by reflection at the front of the cornea, and at the front and back of the crystalline, have shown that the convexity of the front of the crystalline is materially changed as the patient adapts his eye to near or remote vision, the convexity being greatest for near vision. This increase of convexity corresponds to a shortening of focal length, and is thus consistent with theory.

1024. Binocular Vision.—The difficulty which some persons have felt in reconciling the fact of an inverted image on the retina with the perception of an object in its true position, is altogether fanciful, and arises from confused notions as to the nature of perception.

The question as to how it is that we see objects single with two eyes, rests upon a different footing, and is not to be altogether explained by habit and association.[1] To each point in the retina of one eye there is a *corresponding point*, similarly situated, in the other. An impression produced on one of these points is, in ordinary circumstances, undistinguishable from a similar impression produced on the other, and when both at once are similarly impressed, the effect is simply more intense than if one were impressed alone; or, to describe the same phenomena subjectively, we have only one field

[1] Binocular vision is a subject which has been much debated. For the account here given of it, the Editor is responsible.

of view for our two eyes, and in any part of this field of view we see either one image, brighter than we should see it by one alone, or else we see two overlapping images. This latter phenomenon can be readily illustrated by holding up a finger between one's eyes and a wall, and looking at the wall. We shall see, as it were, two transparent fingers projected on the wall. One of these transparent fingers is in fact seen by the right eye, and the other by the left, but our visual sensations do not directly inform us which of them is seen by the right eye, and which by the left.

The principal advantage of having two eyes is in the estimation of distance, and the perception of relief. In order to see a point as single by two eyes, we must make its two images fall on corresponding points of the retinæ; and this implies a greater or less convergence of the optic axes according as the object is nearer or more remote. We are thus furnished with a direct indication of the distance of the object from our eyes; and this indication is much more precise than that derived from the adjustment of their focal length.

In judging of the comparative distances of two points which lie nearly in the same direction, we are greatly aided by the parallactic displacement which occurs when we change our own position.

We can also form an estimate of the nearness of an object, from the amount of change in its apparent size, contour, and bearing, produced by shifting our position. This would seem to be the readiest means by which very young animals can distinguish near from remote objects.

1025. Stereoscope.—The perception of relief is closely connected with the doubleness of vision which occurs when the images on corresponding portions of the two retinæ are not similar. In surveying an object we run our eyes rapidly over its surface, in such a way as always to attain single vision of the particular point to which our attention is for the instant directed. We at the same time receive a somewhat indistinct impression of all the points within our field of view; an impression which, when carefully analysed, is found to involve a large amount of doubleness. These various impressions combine to give us the perception of relief; that is to say, of *form in three dimensions.*

The perception of relief in binocular vision is admirably illustrated by the *stereoscope,* an instrument which was invented by Wheatstone, and reduced to its present more convenient form by Brewster. Two figures are drawn, as in Fig. 735, being perspective representations of

the same object from two neighbouring points of view, such as might be occupied by the two eyes in looking at the object. Thus if the object be a cube, the right eye will have a fuller view of the right

Fig. 735.—Stereoscopic Pictures.

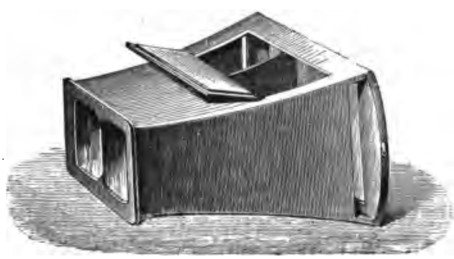


Fig. 736.—Stereoscope.

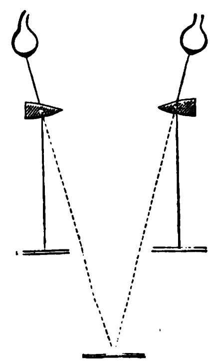


Fig. 737.—Path of Rays in Stereoscope.

face, and the left eye of the left face. The two pictures are placed in the right and left compartments of a box, which has a partition down the centre serving to insure that each eye shall see only the picture intended for it; and over each of the compartments a half-lens is fixed, serving, as in Fig. 737, not only to magnify the picture, but at the same time to displace it, so that the two virtual images are brought into approximate coincidence. Stereoscopic pictures are usually photographs obtained by means of a double camera, having two objectives, one beside the other, which play the part of two eyes.

When matters are properly arranged, the observer seems to see the object in relief. He finds himself able to obtain single view of any one point of the solid image which is before him; and the adjustments of the optic axes which he finds it necessary to make, in shifting his view from one point of it to another, are exactly such as would be required in looking at a solid object.

When one compartment of the stereoscope is empty, and the other contains an object, an observer, of normal vision, looking in in the ordinary way, is unable to say which eye sees the object. If two pictures are combined, consisting of two equal circles, one of them

having a cross in its centre, and the other not, he is unable to decide whether he sees the cross with one eye or both.

When two entirely dissimilar pictures are placed in the two compartments, they compete for mastery, each of them in turn becoming more conspicuous than the other, in spite of any efforts which the observer may make to the contrary. A similar fluctuation will be observed on looking steadily at a real object which is partially hidden from one eye by an intervening object. This tendency to alternate preponderance renders it well nigh impossible to combine two colours by placing one under each eye in the stereoscope.

The immediate visual impression, when we look either at a real solid object, or at the apparently solid object formed by properly combining a pair of stereoscopic views, is a single picture formed of two slightly different pictures superimposed upon each other. The coincidence becomes exact at any point to which attention is directed, and to which the optic axes are accordingly made to converge, but in the greater part of the combined picture there is a want of coincidence, which can easily be detected by a collateral exercise of attention. The fluctuation above described to some extent tends to conceal this doubleness; and in looking at a real solid object, the concealment is further assisted by the blurring of parts which are out of focus.

1026. Visual Angle. Magnifying Power.—The angle which a given straight line subtends at the eye is called its *visual angle*, or the *angle under which it is seen*. This angle is the measure of the length of the image of the straight line on the retina. Two discs at different distances from the eye, are said to have the same apparent size, if their diameters are seen under equal angles. This is the condition that the nearer disc, if interposed between the eye and the remoter disc, should be just large enough to conceal it from view.

The angle under which a given line is seen, evidently depends not only on its real length, and the direction in which it points, but also on its distance from the eye; and varies, in the case of small visual angles, in the inverse ratio of this distance. The *apparent length* of a straight line may be regarded as measured by the visual angle which it subtends.

By the *magnifying power* of an optical instrument, is usually meant the ratio in which it increases *apparent lengths* in this sense. In the case of telescopes, the comparison is between an object as

seen in the telescope, and the same object as seen with the naked eye at its actual distance. In the case of microscopes, the comparison is between the object as seen in the instrument, and the same object as seen by the naked eye at the least distance of distinct vision, which is usually assumed as 10 inches.

But two discs, whose diameters subtend the same angle at the eye, may be said to have the same *apparent area;* and since the areas of similar figures are as the squares of their linear dimensions, it is evident that the apparent area of an object varies as the square of the visual angle subtended by its diameter. The number expressing *magnification of apparent area* is therefore the square of the magnifying power as above defined. Frequently, in order to show that the comparison is not between apparent areas, but between apparent lengths, an instrument is said to magnify so many *diameters.* If the diameter of a sphere subtends 1° as seen by the naked eye, and 10° as seen in a telescope, the telescope is said to have a magnifying power of 10 diameters. The superficial magnification in this case is evidently 100.

The apparent length and apparent area of an object are respectively proportional to the length and area of its image on the retina.

Apparent length is measured by the plane angle, and apparent area by the solid angle, which an object subtends at the eye.

1027. Spectacles.—Spectacles are of two kinds, intended to remedy two opposite defects of vision. Short-sighted persons can see objects distinctly at a smaller distance than persons whose vision is normal; but always see distant objects confused. On the other hand, persons whose vision is normal in their youth, usually become over-sighted with advancing years, so that, while they can still adjust their eyes correctly for distant vision, objects as near as 10 or 12 inches always appear blurred. Spectacles for over-sighted persons are convex, and should be of such focal length, that, when an object is held at about 10 inches distance, its virtual image is formed at the nearest distance of distinct vision for the person who is to use them. This latter distance must be ascertained by trial. Call it p inches; then, by § 1012, the formula for computing the required focal length x (in inches) is

$$\frac{1}{10} - \frac{1}{p} = \frac{1}{x}.$$

For example, if 15 inches is the nearest distance at which the person

can conveniently read without spectacles, the focal length required is 30 inches.

In Fig. 738, A represents the position of a small object, and A′ that of its virtual image as seen with spectacles of this kind.

Over-sight is not the only defect which the eye is liable to acquire

Fig. 738.—Spectacle-glass for Over-sighted Eye.

by age, but it is the defect which ordinary spectacles are designed to remedy.

Spectacles for short-sighted persons are concave, and the focal

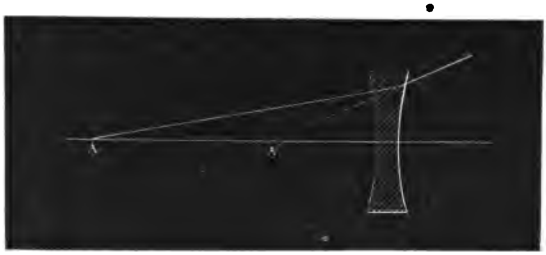


Fig. 739.—Spectacle-glass for Short-sighted Eye.

length which they ought to have, if designed for reading, may be computed by the formula

$$\frac{1}{p} - \frac{1}{10} = \frac{1}{x},$$

p denoting the nearest distance at which the person can read, and x the focal length, both in inches. If his *greatest* distance of distinct vision exceeds the focal length, he will be able, by means of the spectacles, to obtain distinct vision of objects at all distances, from 10 inches upwards.

1028. Simple Magnifier.—A *magnifying glass* is a convex lens, of

shorter focal length than the human eye, and is placed at a distance somewhat less than its focal length from the object to be viewed. In Fig. 740, *ab* is the object, and A B the virtual image which is seen by the eye K. The construction which we have employed for drawing the image is one which we have several times used before. Through the point *a*, the line *a* M is drawn parallel to the principal axis. F M is then drawn from the principal focus F; O *a* is drawn from the optical centre O; and these two lines are produced till they meet in A.

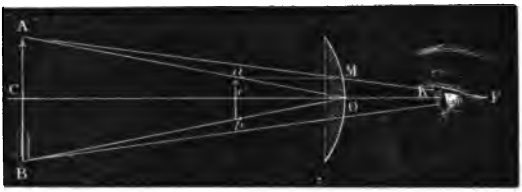


Fig. 740.—Magnifying Glass.

Distance of lens from object. In order that the image may be properly seen, its distance from the eye must fall between the limits of distinct vision; and in order that it may be seen under the largest possible visual angle, the eye must be close to the lens, and the object must be as near as is compatible with distinct vision. This and other interesting properties are established by the following investigation:—

Let θ denote the visual angle under which the observer sees the image of the portion $a\,c$ of the object. Also let x denote the distance c O of the object from the lens, and y the distance O K of the lens from the eye. Then we have

$$\tan\theta = \frac{\mathrm{AC}}{\mathrm{CK}} = \frac{\mathrm{AC}}{\mathrm{CO}+y};$$

but, by formulæ (10) and (11) of last chapter, we have

$$\mathrm{AC} = a\,c\frac{f}{f-x}, \quad \mathrm{CO} = x\,.\frac{f}{f-x}.$$

Substituting these values for A C and C O, and reducing, we have

$$\tan\theta = a\,c\,.\frac{f}{(x+y)\,f - xy}. \qquad \text{(A)}$$

This equation shows that, for a given lens and a given object, the visual angle varies inversely as the quantity $(x+y)\,f - xy$.

The following practical consequences are easily drawn:—

(1) If the distance $x+y$ of the eye from the object is given, the visual angle increases as the two distances x, y approach equality, and is not altered by interchanging them.

(2) If one of the two distances x, y be given, and be less than f, the other must be made as small as possible, if we wish to obtain the largest possible visual angle.

To obtain the absolute maximum of visual angle, we must select, from the various positions which make C K equal to the nearest distance of distinct vision, that which gives the largest value of A C, since the quotient of A C by C K is the tangent of the visual angle. Now A C increases as the image moves further from the lens, and hence the absolute maximum is obtained by making its distance from the lens equal to the nearest distance of distinct vision, and making the eye come up close to the lens. In this case the distance p of the object from the lens is given by the equation $\frac{1}{p} - \frac{1}{D} = \frac{1}{f}$, where D denotes the nearest distance of distinct vision; and $\tan\theta$ is $\frac{ac}{p}$ or $ac\left(\frac{1}{f}+\frac{1}{D}\right)$. But the greatest angle under which the body could be seen by the naked eye is the angle whose tangent is $\frac{ac}{D}$; hence the visual angle (or its tangent) is increased by the lens in the ratio $1+\frac{D}{f}$, which is called the *magnifying power*. If the object were in the principal focus, and the eye close to the lens, the magnifying power would be $\frac{D}{f}$. In either case, the thickness of the lens being neglected, the visual angle is the angle which the object subtends at the centre of the lens, and therefore varies inversely as the distance of this centre from the object. For lenses of small focal length, the reciprocal of the focal length may be regarded as proportional to the magnifying power.

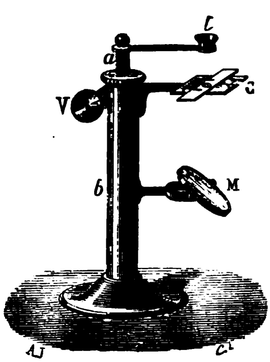


Fig. 741.—Simple Microscope.

Simple Microscope.—By a *simple microscope* is usually understood a lens of short focal length mounted in a manner convenient for the examination of small objects. Fig. 741 represents an instrument of this kind. The lens l is mounted in brass, and carried at the end of an arm. It is raised and lowered by turning the milled head V,

which acts on the rack *a*. C is the platform on which the object is laid, and M is a concave mirror, which can be employed for increasing the illumination of the object.

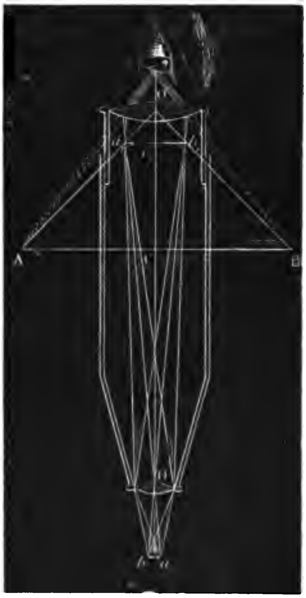


Fig. 742.—Compound Microscope.

1029. Compound Microscope.—In the compound microscope, there is one lens which forms a real and greatly enlarged image of the object; and this image is itself magnified by viewing it through another lens

In Fig. 742, $a\,b$ is the object, O is the first lens, called the *objective*, and is placed at a distance only slightly exceeding its focal length from the object; an inverted image $a_1\,b_1$ is thus formed at a much greater distance on the other side of the lens, and proportionally larger. O′ is the second lens, called the *ocular* or *eye-piece*, which is placed at a distance a little less than its focal length from the first image $a_1\,b_1$, and thus forms an enlarged virtual image of it A B, at a convenient distance for distinct vision.

If we suppose the final image A B to be at the least distance of distinct vision from the eye placed at O′ (this being the arrangement which gives the largest visual angle), the magnifying power will be simply the ratio of the length of this image to that of the object $a\,b$, and will be the product of the two factors $\frac{A\,B}{a_1\,b_1}$ and $\frac{a_1\,b_1}{a\,b}$. The former is the magnification produced by the eye-piece, and is, as we have just shown (§ 1028), $1+\frac{D}{f}$. The other factor $\frac{a_1\,b_1}{a\,b}$ is the magnification produced by the objective, and is equal to the ratio of the distances $\frac{O\,a_1}{O\,a}$. If the objective is taken out, and replaced by another of different focal length, the readjustment will consist in altering the distance O a, leaving the distance O a_1 unchanged. The total magnification therefore varies inversely as O a, that is, nearly in the inverse ratio of the focal length of the objective. Compound microscopes are usually provided with several objectives, of various focal lengths, from which the observer makes a selection according to the magnifying power which he requires for the object to be examined. The powers most used range from 50 to 350 diameters.

The magnifying power of a microscope can be determined by direct observation, in the following way. A plane reflector pierced with a hole in its centre, is placed directly over the eye-piece (Fig. 743), at an inclination of 45°, and another plane reflector, or still better, a totally reflecting prism, as in the figure, is placed parallel to it at the distance of an inch or two, so that the eye, looking down upon the first mirror, sees, by means of two successive reflections, the image of a divided scale placed at a distance of 8 or 10 inches below the second reflector. In taking an observation, a micrometer scale engraved on glass, its divisions being at a known distance apart (say $\frac{1}{100}$ of a millimetre), is placed in the microscope as the object to be magnified; and the observer holds his eye in such a position that, by means of different parts of his pupil, he sees at once the magnified image of the micrometer scale in the microscope, and the reflected and unmagnified image of the other scale. The two images will be superimposed in the same field of view; and it is easy to observe how many divisions of the one coincide with a given number of divisions of the other. Let the divisions on the large scale be millimetres, and those on the micrometer scale hundredths of a millimetre. Then the magnifying power is 100, if one of the magnified covers one of the unmagnified divisions; and is $\frac{100\,N}{n}$, if n of the former cover N of the latter. This is on the assumption that the large scale is placed at the nearest distance of convenient vision. In stating the magnifying power of a microscope, this distance is usually reckoned as 10 inches.

Fig. 743.
Measurement of Magnifying Power.

A short-sighted person sees an image in a microscope (whether simple or compound) under a larger visual angle than a person of normal sight; but the inequality is not so great as in the case of objects seen by the naked eye. In fact, if f be the focal length of the eye-piece in a compound microscope, or of the microscope itself if simple, and D the nearest distance of distinct vision for the

observer, the visual angle under which the image is seen in the microscope is proportional to $\frac{1}{f} + \frac{1}{D}$, the greatest visual angle for the naked eye being represented by $\frac{1}{D}$. Both these angles increase as D diminishes, but the latter increases in a greater ratio than the former. When f is as small as $\frac{1}{10}$ of an inch, the visual angle in the microscope is sensibly the same for short as for normal sight.

Before reading off the divisions in the observation above described, care should be taken to focus the microscope in such a way, that the image of the micrometer scale is at the same distance from the eye as the image of the large scale with which it is compared. When this is done, a slight motion of the eye does not displace one image with respect to the other.

Instead of a single eye-lens, it is usual to employ two lenses separated by an interval, that which is next the eye being called the *eye-glass*, and the other the *field-glass*. This combination is equivalent to the Huygenian or negative eye-piece employed in telescopes (§ 1070).

1030. Astronomical Telescope.—The astronomical refracting telescope consists essentially (like the compound microscope) of two lenses, one of which forms a real and inverted image of the object, which is looked at through the other.

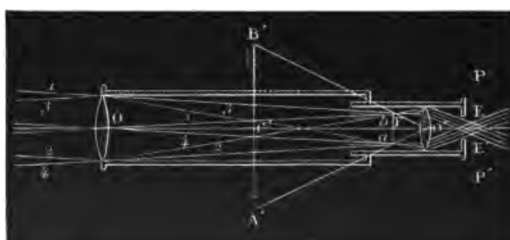


Fig. 744.—Astronomical Telescope.

In Fig. 744, O is the object-glass, which is sometimes a foot or more in diameter, and is always of much greater focal length than the eye-piece O′. The inverted image of a distant object is formed at the principal focus F. This image is represented at $a\,b$. The parallel rays marked 1, 2 come from the upper extremity of the object, and meet at a; and the parallel rays 3, 4, from the other extremity, meet at b. A′ B′ is the virtual image of $a\,b$ formed by the eye-piece. Its distance from the eye can be changed by pulling out or pushing in the eye-tube; and may in practice have any value intermediate between the least distance of distinct vision and infinity, the visual angle under which it is seen being but slightly affected by

this adjustment. The rays from the highest point of the object emerge from the eye-piece as a pencil diverging from A′; and the rays from the lowest point of the object form a pencil diverging from B′.

Magnification.—The angle under which the object would be seen by the naked eye is $a\,O\,b$; for the rays $a\,O, b\,O$, if produced, would pass through its extremities. The angle under which it is seen in the telescope, if the eye be close to the eye-lens, is A′ O′ B′ or $a\,O'\,b$.

The magnification is therefore $\frac{a\,O'\,b}{a\,O\,b}$, which is approximately the same as the ratio of the distances of the image ab from the two lenses $\frac{\text{O F}}{\text{O}'\text{F}}$. If the eye-tube is so adjusted as to throw the image A′ B′ to infinite distance, F will be the principal focus of both lenses, and the magnification is the ratio of the focal length of the object-glass to that of the eye-piece. If the eye-tube be pushed in as far as is compatible with distinct vision (the eye being close to the lens), the magnification is greater than this in the ratio $\frac{D+f}{D}$, D denoting the nearest distance of distinct vision, and f the focal length of the eye-piece

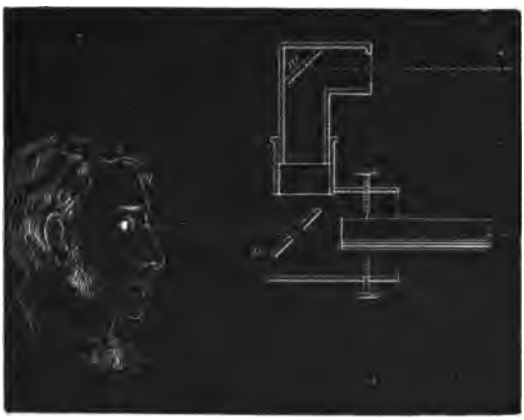


Fig. 745.—Measurement of Magnifying Power.

The magnification can be directly observed by looking with one eye through the telescope at a brick wall, while the other eye is kept open. The image will thus be superimposed on the actual wall, and we have only to observe how many courses of the latter coincide with a single course of the magnified image.

If the telescope is large, its tube may prevent the second eye from seeing the wall, and it may be necessary to employ a reflecting arrangement, as in Fig. 745, analogous to that described in connection with the microscope.

Telescopes without stands seldom magnify more than about 10 diameters. Powers of from 20 to 60 are common in telescopes with

stands, intended for terrestrial purposes. The powers chiefly employed in astronomical observation are from 100 to 500.

Mechanical Arrangements.—The achromatic object-glass O is set in a mounting which is screwed into one end of a strong brass tube A A (Fig. 746). In the other end slides a smaller tube F containing the eye-piece O′; and by turning the milled head V in one direction or the other, the eye-piece is moved forwards or backwards.

Fig. 746.—Astronomical Telescope.

Finder.—The small telescope *l*, which is attached to the principal telescope, is called a *finder.* This appendage is indispensable when the principal telescope has a high magnifying power; for a high magnifying power involves a small field of view, and consequent difficulty in directing the telescope so as to include a selected object within its range. The finder is a telescope of large field; and as it is set parallel to the principal telescope, objects will be visible in the latter if they are seen in the centre of the field of view of the former.

1031. Best Position for the Eye.—The eye-piece forms a real and inverted image of the object-glass[1] at E E′ (Fig. 744), through which all rays transmitted by the telescope must of necessity pass. If the telescope be directed to a bright sky, and a piece of white paper held behind the eye-piece to serve as a screen, a circular spot of light will be formed upon it, which will become sharply defined (and at the same time attain its smallest size) when the screen is held in the correct position. This image (which we shall call the *bright spot*) may be regarded as marking the proper place for the pupil of the observer's eye. Every ray which traverses the centre of the object-glass traverses the centre of this spot; every ray which traverses the upper edge of the object-glass traverses the lower edge of the

[1] Or it may be called an image of *the aperture which the object-glass fills.* It remains sensibly unchanged on removing the object-glass so as to leave the end of the telescope open.

spot; and any selected point of the spot receives all the rays which have been transmitted by one particular point of the object-glass. An eye with its pupil anywhere within the limits of the bright spot, will therefore see the whole field of view of the telescope. If the spot and pupil are of exactly the same size, they must be made to coincide with one another, as the necessary condition of seeing the whole field of view with the brightest possible illumination. Usually in practice the spot is much smaller than the pupil, so that these advantages can be obtained without any nicety of adjustment; but to obtain the most distinct vision, the centre of the pupil should coincide as closely as possible with the centre of the spot. To facilitate this adjustment, a brass diaphragm, with a hole in its centre, is screwed into the eye-end of the telescope, the proper place for the eye being close to this hole.

One method of determining the magnifying power of a telescope consists in measuring the diameter of the bright spot, and comparing it with the effective aperture of the object-glass. In fact, let F and f denote the focal lengths of object-glass and eye-piece, and a the distance of the spot from the centre of the eye-piece; then $F+f$ is approximately the distance of the object-glass from the same centre, and, by the formula for conjugate focal distances, we have $\frac{1}{F+f}+\frac{1}{a}=\frac{1}{f}$. Multiplying both sides of this equation by $F+f$, and then subtracting unity, we have $\frac{F+f}{a}=\frac{F}{f}$. But the ratio of the diameter of the object-glass to that of its image is $\frac{F+f}{a}$; and $\frac{F}{f}$ is the usual formula for the magnifying power. Hence, *the linear magnifying power of a telescope is the ratio of the diameter of the object-glass to that of the bright spot.*

1032. Terrestrial Telescope.—The astronomical telescope just described gives inverted images. This is no drawback in astronomical observation, but would be inconvenient in viewing terrestrial objects. In order to re-invert the image, and thus make it erect, two additional lenses O″ O‴ (Fig. 747) are introduced between the real image $a\,b$ and the eye-lens O′. If the first of these O″ is at the distance of its principal focal length from $a\,b$, the pencils which fall upon the second will be parallel, and an erect image $a'\,b'$ will thus be formed in the principal focus of O‴. This image is viewed through the eye-lens O′, and the virtual image A′ B′ which is perceived by the eye will therefore be erect. The two lenses O″, O‴, are usually

made precisely alike, in which case the two images *a b*, *a′ b′* will be equal. In the better class of terrestrial telescopes, a different ar-

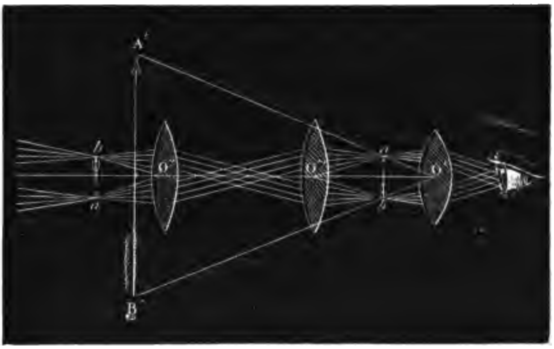


Fig. 747.—Terrestrial Eye-piece.

rangement is adopted, requiring one more lens; but whatever system be employed, the reinversion of the image always involves some loss both of light and of distinctness.

1033. Galilean Telescope.—Besides the disadvantages just mentioned, the erecting eye-piece involves a considerable addition to the length of the instrument. The telescope invented by Galileo, and the earliest of all telescopes, gives erect images with only two lenses, and with shorter length than even the astronomical telescope. O (Fig. 748) is the object-glass, which is convex as in the astronomical telescope, and would form a real and inverted image *a b* at its principal focus; but the eye-glass O′, which is a concave lens, is interposed at a distance equal to or slightly exceeding its own focal length from the place of this image, and forms an erect virtual image A′ B′, which the observer sees.

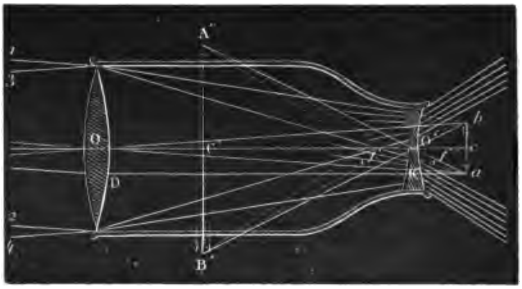


Fig. 748.—Galilean Telescope.

Neglecting the distance of his eye from the lens, the angle under which he sees the image is A′ O′ B′, which is equal to *a* O′ *b*, whereas

the visual angle to the naked eye would be $a \mathrm{O} b$. The magnification is therefore $\frac{a \mathrm{O}' b}{a \mathrm{O} b}$, which is approximately equal to $\frac{\mathrm{O} c}{\mathrm{O}' c}$, c being the principal focus of the object-glass. If the instrument is focussed in such a way that the image A′ B′ is thrown to infinite distance, c is also the principal focus of the eye-lens, and the magnification is simply the ratio of the focal lengths of the two lenses. This is the same rule which we deduced for the astronomical telescope; but the Galilean telescope, if of the same power, is shorter by twice the focal length of the eye-lens, since the distance between the two lenses is the difference instead of the sum of their focal lengths.

This telescope has the disadvantage of not admitting of the employment of cross wires; for these, in order to serve their purpose, must coincide with the real image; and no such image exists in this telescope.

Fig. 749.– Opera-glass.

There is another peculiarity in the absence of the *bright spot* above described, the image of the object-glass formed by the eye-glass being virtual. In other telescopes, if half the object-glass be covered, half the bright spot will be obliterated; but the remaining half suffices for giving the whole field of view, though with diminished brightness. In the Galilean telescope, on the contrary, if half the object-glass be covered, half the field of view will be cut off, and the remaining half will be unaffected.

The *opera-glass*, single or binocular, is a Galilean telescope, or a pair of Galilean telescopes. In the best instruments, both object-glass and eye-glass are achromatic combinations of three pieces, as shown in section in the figure (Fig. 749); the middle piece in each case being flint, and the other two crown (§ 1064).

1034. Reflecting Telescopes.—In reflecting telescopes, the place of an object-glass is supplied by a concave mirror called a *speculum*, usually composed of solid metal. The real and inverted image which it forms of distant objects is, in the Herschelian telescope, viewed directly through an eye-piece, the back of the observer being towards the object, and his face towards the speculum. This construction is only applicable to very large specula; as in instruments of ordinary

size the interposition of the observer's head would occasion too serious a loss of light.

An arrangement more frequently adopted is that devised by Sir Isaac Newton, and employed by him in the first reflecting telescope ever constructed. It is represented in Fig. 750. The speculum is at the bottom of a tube whose open end is directed towards the distant object to be examined. The rays 1 and 2 from one extremity of the object are reflected towards a, and the rays 3, 4 from the other extremity are reflected towards b. A real inverted image $a\,b$ would thus be formed at the principal focus of the concave speculum; but a small plane mirror M is interposed obliquely, and causes the real image to be formed at $a'\,b'$ in a symmetrical position with respect to the mirror M. The eye-lens O′ transforms this into the enlarged and virtual image A′ B′.

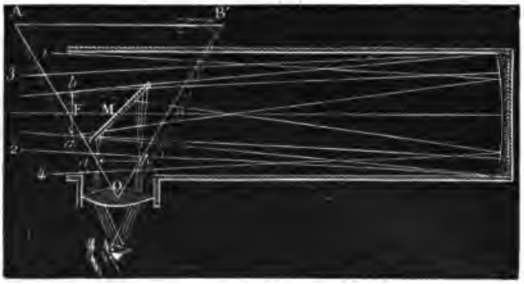


Fig. 750.—Newtonian Telescope.

Magnifying Power.—The approximate formula for the magnifying power is the same as in the case of the refracting telescopes already described. In fact the first image $a\,b$ subtends, at the optical centre O (not shown in the figure) of the large speculum, an angle $a\,O\,b$ equal to the visual angle for the naked eye; and the second image $a'\,b'$ (which is equal to the former) subtends, at the centre of the eye-piece, an angle $a'\,O'\,b'$ equal to the angle under which the image is seen in the telescope. The magnifying power is therefore $\frac{a'\,O'\,b'}{a\,O\,b}$, or, what is the same thing, is the ratio of the distance of $a\,b$ from O to the distance of $a'\,b'$ from O′, or the ratio of the focal length of the speculum to that of the eye-piece.

In the Gregorian telescope, which was invented before that of Newton, but not manufactured till a later date, there are two concave specula. The large one, which receives the direct rays from the object, forms a real and inverted image. The smaller speculum which is suspended in the centre of the tube, with its back to the object, receives the rays reflected from the first speculum, and forms a second real image, which is the enlarged and inverted image of the

first, and is therefore erect as compared with the object. This real and erect image is then magnified by means of an eye-piece, as in the instruments previously described, the eye-piece being contained in a tube which slides in a hole pierced in the middle of the large speculum.

As this arrangement gives an erect image, and enables the observer to look directly towards the object, it is specially convenient for terrestrial observation. It is the construction almost universally adopted in reflecting telescopes of small size.

The Cassegranian telescope resembles the Gregorian, except that the second speculum is convex, and the image which the observer sees is inverted.

1035. Silvered Specula.—Achromatic refracting telescopes give much better results, both as regards light and definition, than reflectors of the same size or weight; but it has been found practicable to make specula of much larger size than object-glasses. The aperture of Lord Rosse's largest telescope is 6 feet, whereas the aperture of the largest achromatic telescopes yet constructed is less than two feet, and increase of size involves increased thickness of glass, and consequent absorption of light.

The massiveness which is found necessary in the speculum in order to prevent flexure, is a serious inconvenience, as is also the necessity for frequent repolishing—an operation of great delicacy, as the slightest change in the form of the surface impairs the definition of the images. Both these defects have been to a certain extent remedied by the introduction of glass specula, covered in front with a thin coating of silver. Glass is much more easily worked than speculum-metal (which is remarkable for its brittleness in casting), and has only one-third of its specific gravity. Silver is also much superior to speculum-metal in reflecting power; and as often as it becomes tarnished it can be removed and renewed, without liability to change of form in the reflecting surface.[1]

1036. Measure of Brightness.—The apparent brightness of a surface is most naturally measured by the amount of light per unit area of its image on the retina; and therefore varies *directly as the amount of light which the surface sends to the pupil, and inversely as the apparent area of the surface.*

To avoid complications arising from the varying condition of the

[1] The merits of silvered specula are fully set forth in a brochure published by Mr. Browning, the optician, entitled *A Plea for Reflectors.*

observer, we shall leave dilatation and contraction of the pupil out of account.

When a body is looked at through a small pinhole in a card held close to the eye, it appears much darker than when viewed in the ordinary way; and in like manner images formed by optical instruments often furnish beams of light too narrow to fill the pupil. In all such cases it becomes necessary to distinguish between *effective brightness* and *intrinsic brightness*, the former being less than the latter in the same ratio in which the cross section of the beam which enters the pupil is less than the area of the pupil. We may correctly speak of the *intrinsic brightness* of a surface *for a particular point* of the pupil; and the effective brightness will in every case be the average value of the intrinsic brightness taken over the whole pupil.

In the case of natural bodies viewed in the ordinary way, the distinction may be neglected, since they usually send light in sensibly equal amounts to all parts of the pupil.

To obtain a numerical measure of intrinsic brightness, let us denote by s a small area on a surface directly facing towards the eye, or the foreshortened projection of a small area oblique to the line of vision, and by ω the solid angle which the pupil of the eye subtends at any point of s. Then the quantity of light q which s sends to the pupil per unit time, varies jointly as the solid angle ω, the area s, and the intrinsic brightness of s, which we will denote by I. We may therefore write

$$q = \mathrm{I}\, s\, \omega, \text{ and } \mathrm{I} = \frac{q}{s\,\omega}.$$

The intrinsic brightness of a small area s is therefore measured by $\frac{q}{s\,\omega}$, where q denotes the quantity of light which s emits per unit time in directions limited by the small angle of divergence ω.

1037. Applications.—One of the most obvious consequences is that *surfaces appear equally bright at all distances* in the same direction, provided that no light is stopped by the air or other intervening medium; for q and ω both vary inversely as the square of the distance. The area of the image formed on the retina in fact varies directly as the amount of light by which it is formed.

Images formed by Lenses.—Let A B (Fig. 751) be an object, and $a\,b$ its real image formed by the lens C D, whose centre is O. Let S denote a small area at A, and Q the quantity of light which it sends to the lens; also let s denote the corresponding area of the

image, and q the quantity of light which traverses it. Then q would be identical with Q if no light were stopped by the lens; the areas, S, s, are directly as the squares of the conjugate focal distances O A, O a; and the solid angles of divergence Ω and ω, for Q and q, being the solid angles subtended by the lens at A and a (for the plane angle $c\,a\,d$ in the figure is equal to the vertical angle $C\,a\,D$), are inversely as the squares of the conjugate focal distances. We have accordingly $S\,\Omega = s\,\omega$. The intrinsic brightness $\frac{q}{s\,\omega}$ of the image is therefore equal to the intrinsic brightness $\frac{Q}{S\,\Omega}$ of the object except in so far as light is stopped by the lens. Precisely similar reasoning applies to virtual images formed by lenses.[1]

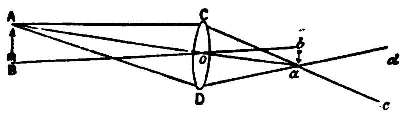


Fig. 751.—Brightness of Image.

In the case of images formed by mirrors, Ω and ω are the solid angles subtended by the mirror at the conjugate foci, and are inversely as the squares of the distances from the mirror; while S and s are directly as the squares of the distances from the centre of curvature; but these four distances are proportional (§ 967), so that the same reasoning is still applicable. If the mirror only reflects half the incident light, the image will have only half the intrinsic brightness of the object.

If the pupil is filled with light from the image, the effective brightness will be the same as the intrinsic brightness thus computed. If this condition is not fulfilled, the former will be less than the latter. When the image is greatly magnified as compared with the object, the angle of divergence is greatly diminished in comparison with the angle which the lens or mirror subtends at the object, and often becomes so small that only a small part of the pupil is utilized. This is the explanation of the great falling off of light which is observed in the use of high magnifying powers, both in microscopes and telescopes.

[1] For refraction out of a medium of index μ_1 into another of index μ_2, we have by § 1015, equation (13), $\mu_1 : \mu_2 :: \frac{A\,P_1}{C\,P_1} : \frac{A\,P_2}{C\,P_2}$. But since s_1, s_2 are the areas of corresponding parts of object and image, we have $s_1 : s_2 :: C\,P_1^2 : C\,P_2^2$, and since ω_1, ω_2 are the solid angles subtended at P_1, P_2 by one and the same portion of the bounding surface, we have $\omega_1 : \omega_2 :: A\,P_2^2 : A\,P_1^2$. Therefore $\frac{q}{s_1\,\omega_1} : \frac{q}{s_2\,\omega_2} :: \mu_1^2 : \mu_2^2$. The intrinsic brightnesses of a succession of images in different media are therefore directly as the squares of the absolute indices. On this point see *Phil. Mag.*, March, 1888, p. 216.

1038. Brightness of Image in a Telescope.—It has been already pointed out (§ 1031) that in most forms of telescope (the Galilean being an exception), there is a certain position, a little behind the eye-piece, at which a well-defined bright spot is formed upon a screen held there while the telescope is directed to any distant source of light. It has also been pointed out that this spot is the image, formed by the eye-piece, of the opening which is filled by the object-glass, and that the magnifying power of the instrument is the ratio of the size of the object-glass to the size of this bright spot.

Let s denote the diameter of the bright spot, o the diameter of the object-glass, e the diameter of the pupil of the eye; then $\frac{o}{s}$ is the linear magnifying power.

We shall first consider the case in which the spot exactly covers the pupil of the observer's eye, so that $s=e$. Then the whole light which traverses the telescope from a distant object enters the eye; and if we neglect the light stopped in the telescope, this is the whole light sent by the object to the object-glass, and is $\left(\frac{o}{e}\right)^2$ times that which would be received by the naked eye. The magnification of apparent area is $\left(\frac{o}{s}\right)^2$, which, from the equality of s and e, is the same as the increase of total light. The brightness is therefore the same as to the naked eye.

Next, let s be greater than e, and let the pupil occupy the central part of the spot. Then, since the spot is the image of the object-glass, we may divide the object-glass into two parts—a central part whose image coincides with the pupil, and a circumferential part whose image surrounds the pupil. All rays from the object which traverse the central part, traverse its image, and therefore enter the pupil; whereas rays traversing the circumferential part of the object-glass, traverse the circumferential part of the image, and so are wasted. The area of the central part (whether of the object-glass or of its image) is to the whole area as $e^2 : s^2$; and the light which the object sends to the central portion, instead of being $\left(\frac{o}{e}\right)^2$ times that which would be received by the naked eye, is only $\left(\frac{o}{s}\right)^2$ times. But $\left(\frac{o}{s}\right)^2$ is the magnification of apparent area. Hence the brightness is the same as to the naked eye. In these two cases, effective and intrinsic brightness are the same.

Lastly (and this is by far the most common case in practice), let s be less than e. Then no light is wasted, but the pupil is not filled. The light received is $\left(\frac{o}{e}\right)^2$ times that which the naked eye would receive; and the magnification of apparent area is $\left(\frac{o}{s}\right)^2$. The effective brightness of the image, is to the brightness of the object to the naked eye, as $\left(\frac{o}{e}\right)^2 : \left(\frac{o}{s}\right)^2$; that is, as $s^2 : e^2$; that is, as the area of the bright spot to the whole area of the pupil.

To correct for the light stopped by reflection and imperfect transparency, we have simply to multiply the result in each case by a proper fraction, expressing the ratio of the transmitted to the incident light. This ratio, for the central parts of the field of view, is about 0·85 in the best achromatic telescopes. In such telescopes, therefore, the brightness of the image cannot exceed 0·85 of the brightness of the object to the naked eye. It will have this precise value, when the magnifying power is equal to or less than $\frac{o}{e}$; and from this point upwards will vary inversely as the square of the linear magnification.

The same formulæ apply to reflecting telescopes, o denoting now the diameter of the large speculum which serves as objective; but the constant factor is usually considerably less than 0·85.

It may be accepted as a general principle in optics, that while it is possible, by bad focussing or instrumental imperfections, to obtain a confused image whose brightness shall be intermediate between the brightest and the darkest parts of the object, *it is impossible, by any optical arrangement whatever, to obtain an image whose brightest part shall surpass the brightest part of the object.*

1039. Brightness of Stars.—There is one important case in which the foregoing rules regarding the brightness of images become nugatory. The fixed stars are bodies which subtend at the earth angles smaller than the *minimum visibile*, but which, on account of their excessive brightness, *appear* to have a sensible angular diameter. This is an instance of *irradiation*, a phenomenon manifested by all bodies of excessive brightness, and consisting in an extension of their apparent beyond their actual boundary. What is called, in popular language, a bright star, is a star which sends a large total amount of light to the eye.

Denoting by a the ratio of the transmitted to the whole incident light, a ratio which, as we have seen, is about 0·85 in the most

favourable cases, and calling the light which a star sends to the naked eye unity, the light perceived in its image will be $a\left(\frac{o}{s}\right)^2$, or $a \times$ square of linear magnification, if the bright spot is as large as the pupil. When the eye-piece is changed, increase of power diminishes the size of the spot, and increases the light received by the eye, until the spot is reduced to the size of the pupil. After this, any further magnification has no effect on the quantity of light received, its constant value being $a\left(\frac{o}{e}\right)^2$.

The value of this last expression, or rather the value of $a\,o^2$, is the measure of what is called the *space-penetrating power* of a telescope; that is to say, the power of rendering very faint stars visible; and it is in this respect that telescopes of very large aperture, notably the great reflector of Lord Rosse, are able to display their great superiority over instruments of moderate dimensions.

We have seen that the total light in the visible image of a star remains unaltered, by increase of power in the eye-piece above a certain limit. But the visibility of faint stars in a telescope is promoted by darkening the back-ground of sky on which they are seen. Now the brightness of this back-ground varies directly as s^2, or inversely as the square of the linear magnification (s being supposed less than e). Hence it is advantageous, in examining very faint stars, to employ eye-pieces of sufficient power to render the bright spot much smaller than the pupil of the eye.

1040. Images on a Screen.—Thus far we have been speaking of the brightness of images as viewed directly. Images cast upon a screen are, as a matter of fact, much less brilliant; partly because the screen sends out light in all directions, and therefore through a much larger solid angle than that formed by the beam incident on the screen, and partly because some of the incident light is absorbed.

Let A be the area of the object, which we suppose to face directly towards the lens by which the image is thrown upon the screen, a the area of the image, and D, d their respective distances from the lens. Then if I denote the intrinsic brightness of the object, the light sent from A to the lens will be the product of I A by the solid angle which the lens subtends at the object. This solid angle will be $\frac{\mathrm{L}}{\mathrm{D}^2}$, if L denote the area of the lens. $\mathrm{I\,A}\,\frac{\mathrm{L}}{\mathrm{D}^2}$ is therefore the light sent by the object to the lens, and if we neglect reflection and absorption all this light falls upon the image. *The light which falls*

on unit area of the image is therefore $I\frac{A}{a}\frac{L}{D^2}$, that is $I\frac{L}{d^2}$; it *is* therefore *the same as if the lens were a source of light of brightness I.* Accordingly, if the image of a lamp flame be thrown upon the pupil of an observer's eye, and be large enough to cover the pupil, he will see the lens filled with light of a brightness equal to that of the flame seen directly.

1041. Field of View in Astronomical Telescope.—Let $p\,m\,n\,q$ (Fig. 752) be the common focal plane of the object-glass and eye-glass. Draw Ba, Ab joining the highest points of both, and the lowest points of both; also Aa, Bb joining the highest point of each with the lowest point of the other. Evidently Ba, Ab will be the boundaries of the beam of light transmitted through the telescope, and therefore the points p and q in which these lines intersect the focal plane, will be the extremities of that part of the real image which sends rays to the eye. The angle subtended by $p\,q$ at the centre of the object-glass will therefore be the angular diameter of the complete field of view. But the outer portions of this field will be less bright than the centre, and the full amount of brightness, as calculated in § 1038 for the case in which the "bright spot" is smaller than the pupil, will belong only to the portion $m\,n$ bounded by the cross-lines Aa, Bb; for all the rays sent by the object-glass through the part $m\,n$ traverse the eye-glass, and therefore the bright spot, whereas some of the rays sent by the object-glass to any point between m and p, or between n and q pass wide of the eye-glass and therefore do not reach the bright spot. The complete field of view, as seen by an eye whose pupil includes the bright spot, accordingly consists of a central disc $m\,n$ of full brightness, surrounded by a ring extending to p and q whose brightness gradually diminishes from full brightness at its junction with the disc to zero at its outer boundary. This ring is called the "ragged edge," and is put out of sight in actual telescopes by an opaque stop of annular form in the focal plane. The angular diameter of the field of view, excluding the ragged edge, will be equal to the angle which $m\,n$ subtends at the centre of the object-glass.

Fig. 752.—Field of View.

Fig. 753.—Calculation of Field.

To calculate the length of mn, join D, d, the centres of the object-glass and eye-glass (Fig. 753). The joining line will obviously pass through the intersection of Aa, Bb, and also through the middle point of mn. Draw a parallel to this line through m. Then, by comparing the similar triangles of which am, Am are the hypotenuses, we have

$$ad - mo : od :: \mathrm{AD} + mo : \mathrm{D}o.$$

Hence, multiplying extremes and means, and denoting the focal lengths Do, od by F, f, we have

$$\mathrm{F}(ad - mo) = f(\mathrm{AD} + mo),$$

whence

$$mo = \frac{\mathrm{F}.ad - f.\mathrm{AD}}{\mathrm{F}+f}.$$

This is the radius of the real image, excluding the ragged edge; and the angular radius of the field of view will be

$$\frac{mo}{\mathrm{F}} = \frac{\mathrm{F}.ad - f.\mathrm{AD}}{\mathrm{F}(\mathrm{F}+f)}$$
$$= \frac{ad}{\mathrm{F}+f} - \frac{f.\mathrm{AD}}{\mathrm{F}(\mathrm{F}+f)}.$$

The first term $\frac{ad}{\mathrm{F}+f}$ is the angle which the radius of the eye-glass subtends at the object-glass. But, it is obvious from Fig. 752 that the line aD would bisect mp. Hence the second term represents half the breadth of the ragged edge, and the whole field of view, including the ragged edge, has an angular radius

$$\frac{ad}{\mathrm{F}+f} + \frac{f.\mathrm{AD}}{\mathrm{F}(\mathrm{F}+f)}.$$

1042. Cross-wires of Telescopes.—We have described in § 1010 a mode of marking the place of a real image by means of a cross of threads. When telescopes are employed to assist in the measurement of angles, a contrivance of this kind is almost always introduced. A cross of silkworm threads, in instruments of low power, or of spider threads in instruments of higher power, is stretched across a metallic frame just in front of the eye-piece. The observer must first adjust the eye-piece for distinct vision of this cross, and must then (in the case of theodolites and other surveying instruments) adjust the distance of the object-glass until the object which is to be observed is also seen distinctly in the telescope. The image of the object will then be very nearly in the plane of the cross. If it is not exactly in the plane, parallactic displacement will be observed when the eye is shifted, and this must be cured by slightly

altering the distance of the object-glass. When the adjustment has been completed, the cross always marks one definite point of the object, however the eye be shifted. This coincidence will not be disturbed by pushing in or pulling out the eye-piece; for the frame which carries the cross is attached to the body of the telescope, and the coincidence of the cross with a point of the image is real, so that it could be observed by the naked eye, if the eye-piece were removed. The adjustment of the eye-piece merely serves to give distinct vision, and this will be obtained simultaneously for both the cross and the object.

1043. Line of Collimation.—The employment of *cross-wires* (as these crossing threads are called) enormously increases our power of making accurate observations of direction, and constitutes one of the greatest advantages of modern over ancient instruments.

The line which is regarded as the line of sight, or as the direction in which the telescope is pointed, is called the *line of collimation.* If we neglect the curvature of rays due to atmospheric refraction, we may define it as the *line joining the cross to the object whose image falls on it.* More rigorously, the line of collimation is the *line joining the cross to the optical centre of the object-glass.* When it is desired to adjust the line of collimation,—for example, to make it truly perpendicular to the horizontal axis on which the telescope is mounted, the adjustment is performed by shifting the frame which carries the wires, slow-motion screws being provided for this purpose. Telescopes for astronomical observation are often furnished with a number of parallel wires, crossed by one or two in the transverse direction; and the line of collimation is then defined by reference to an imaginary cross, which is the centre of mean position of all the actual crosses.

1044. Micrometers.—Astronomical micrometers are of various kinds, some of them serving for measuring the angular distance between two points in the same field of view, and others for measuring their apparent direction from one another. They often consist of spider threads placed in the principal focus of the object-glass, so as to be in the same plane as the images of celestial objects, one or more of the threads being movable by means of slow-motion screws, furnished with graduated circles, on which parts of a turn can be read off.

One of the commonest kinds consists of two parallel threads, which can thus be moved to any distance apart, and can also be turned round in their own plane.

CHAPTER LXXII.

DISPERSION. STUDY OF SPECTRA.

1045. Newtonian Experiment.—In the chapter on refraction, we have postponed the discussion of one important phenomenon by which it is usually accompanied, and which we must now proceed to explain. The following experiment, which is due to Sir Isaac Newton, will furnish a fitting introduction to the subject.

On an extensive background of black, let three bright strips be laid in line, as in the left-hand part of Fig. 754, and looked at through a prism with its refracting edge parallel to the strips. We

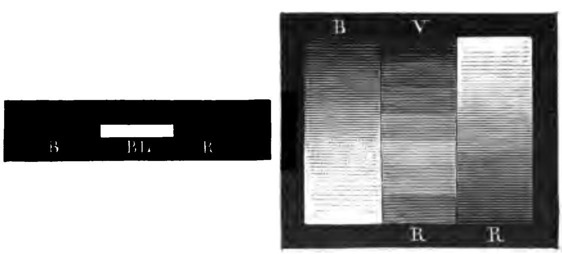


Fig. 754.—Spectra of White and Coloured Strips.

shall suppose the edge to be upward, so that the image is raised above the object. The images, as represented in the right-hand part of Fig. 754, will have the same horizontal dimensions as the strips, but will be greatly extended in the vertical direction; and each image, instead of having the uniform colour of the strips from which it is derived, will be tinted with a gradual succession of colours from top to bottom. Such images are called *spectra*.

If one of the strips (the middle one in the figure) be white, its spectrum will contain the following series of colours, beginning at the top: *violet, blue, green, yellow, orange, red.*

If one of the strips be blue (the left-hand one in the figure), its image will present bright colours at the upper end; and these will be identical with the colours adjacent to them in the spectrum of white. The colours which form the lower part of the spectrum of white will either be very dim and dark in the spectrum of blue, or will be wanting altogether, being replaced by black.

If the other strip be red, its image will contain bright colours at the lower or red end, and those which belong to the upper end of the spectrum of white will be dim or absent. Every colour that occurs in the spectrum of blue or of red will also be found, and in the same horizontal line, in the spectrum of white.

If we employ other colours instead of blue or red, we shall obtain analogous results; every colour will be found to give a spectrum which is identical with part of the spectrum of white, both as regards colour and position, but not generally as regards brightness.

We may occasionally meet with a body whose spectrum consists only of one colour. The petals of some kinds of convolvulus give a spectrum consisting only of blue, and the petals of nasturtium give only red.

1046. Composite Nature of Ordinary Colours.—This experiment shows that the colours presented by the great majority of natural bodies are composite. When a colour is looked at with the naked eye, the sensation experienced is the joint effect of the various elementary colours which compose it. The prism serves to resolve the colour into its components, and exhibit them separately. The experiment also shows that a mixture of all the elementary colours in proper proportions produces white.

1047. Solar Spectrum.—The coloured strips in the foregoing experiment may be illuminated either by daylight or by any of the ordinary sources of artificial light. The former is the best, as gas-light and candle-light are very deficient in blue and violet rays.

Colour, regarded as a property of a coloured (opaque) body, is the power of selecting certain rays and reflecting them either exclusively or in larger proportion than others. The spectrum presented by a body viewed by reflected light, as ordinary bodies are, can thus only consist of the rays, or a selection of the rays, by which the body is illuminated.

A beam of solar light can be directly resolved into its constituents by the following experiment, which is also due to Newton, and was the first demonstration of the composite character of solar light.

Let a beam of sun-light be admitted through a small opening into a dark room. If allowed to fall normally on a white screen, it produces (§ 938) a round white spot, which is an image of the sun. Now let a prism be placed in its path edge-downwards, as in Fig. 755; the

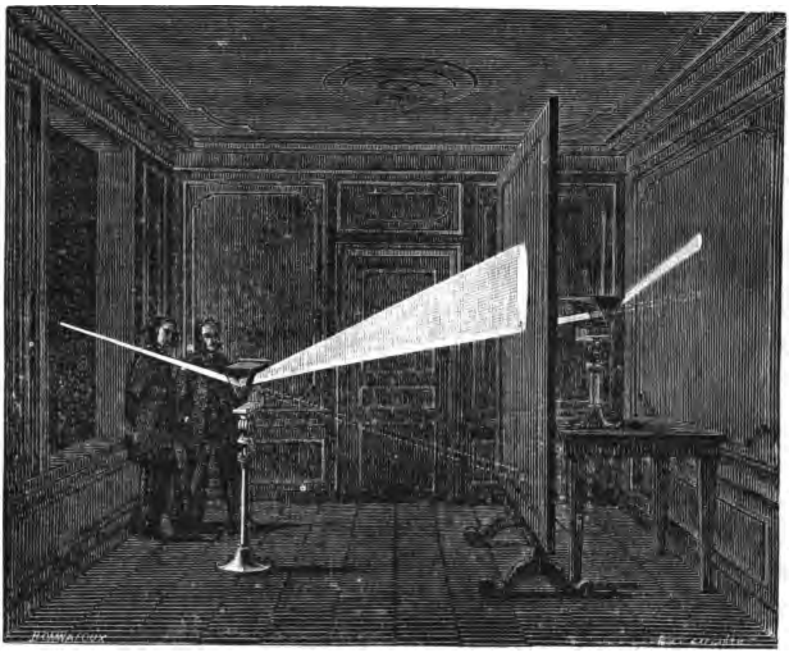


Fig. 755.—Solar Spectrum by Newton's Method.

beam will thus be deflected upwards, and at the same time resolved into its component colours. The image depicted on the screen will be a many-coloured band, resembling the spectrum of white described in § 1045. It will be of uniform width, and rounded off at the ends, being in fact built up of a number of overlapping discs, one for each kind of elementary ray. It is called the *solar spectrum.*

The rays which have undergone the greatest deviation are the violet. They occupy the upper end of the spectrum in the figure. Those which have undergone the least deviation are the red. Of all visible rays, the violet are the most, and the red the least refrangible; and the analysis of light into its components by means of the prism is due to difference of refrangibility. If a small opening is made in the screen, so as to allow rays of only one colour to pass, it will be found,

on transmitting these through a second prism behind the screen, as in Fig. 755, that no further analysis can be effected, and the whole of the image formed by receiving this transmitted light on a second screen will be of this one colour.

1048. Mode of obtaining a Pure Spectrum.—The spectra obtained by the methods above described are built up of a number of overlapping images of different colours. To prevent this overlapping, and obtain each elementary colour pure from all admixture with the rest, we must in the first place employ as the object for yielding the images a very narrow line; and in the second place we must take care that the images which we obtain of this line are not blurred, but have the greatest possible sharpness. A spectrum possessing these characteristics is called pure.

The simplest mode of obtaining a pure spectrum consists in looking through a prism at a fine slit in the shutter of a dark room. The edges of the prism must be parallel to the slit, and its distance from the slit should be five feet or upwards. The observer, placing his eye close to the prism, will see a spectrum; and he should rotate the prism on its axis until he has brought this spectrum to its smallest angular distance from the real slit, of which it is the image.

Let E (Fig. 756) be the position of the eye, S that of the slit. Then the extreme red and violet images of the slit will be seen at R, V, at distances from the prism sensibly equal to the real distance of S (§ 997); and the other images, which compose the remainder of the spectrum, will occupy positions between R and V. The spectrum, in this mode of operating, is virtual.

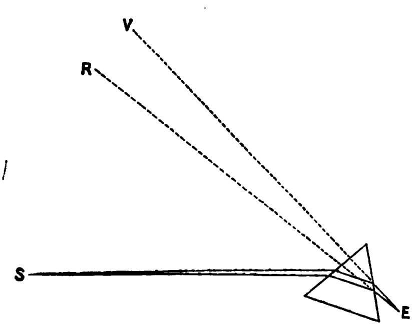


Fig. 756.—Arrangement for seeing a Pure Spectrum.

To obtain a real spectrum in a state of purity, a convex lens must be employed. Let the lens L (Fig. 757) be first placed in such a position as to throw a sharp image of the slit S upon a screen at I. Next let a prism P be introduced between the lens and screen, and rotated on its axis till the position of minimum deviation is obtained, as shown by the movements of the impure spectrum which travels about the walls of the room. Then if the screen be moved into the position R V, its distance from the prism being the

same as before, a pure spectrum will be depicted upon it. A similar result can be obtained by placing the prism between the lens and the slit, but the adjustments are rather more troublesome. Direct

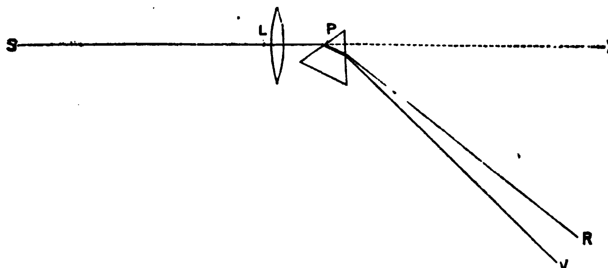


Fig. 757.—Arrangement for Pure Spectrum on Screen.

sun-light, or sun-light reflected from a mirror placed outside the shutter, is necessary for this experiment, as sky-light is not sufficiently powerful. It is usual, in experiments of this kind, to employ a movable mirror called a *heliostat*, by means of which the light can be reflected in any required direction. Sometimes the movements of the mirror are obtained by hand; sometimes by an ingenious clock-work arrangement, which causes the reflected beam to keep its direction unchanged notwithstanding the progress of the sun through the heavens.

The advantage of placing the prism in the position of minimum deviation is twofold. First, the adjustments are facilitated by the equality of conjugate focal distances, which subsists in this case and in this only. Secondly and chiefly, this is the only position in which the images are not blurred. In any other position it can be shown[1] that a small cone of homogeneous incident rays is no longer a cone (that is, its rays do not accurately pass through one point) after transmission through the prism.

The method of observation just described was employed by Wollaston, in the earliest observations of a pure spectrum ever obtained. Fraunhofer, a few years later, independently devised the same method, and carried it to much greater perfection. Instead of looking at the virtual image with the naked eye, he viewed it through a telescope, which greatly magnified it, and revealed several features never before detected. The prism and telescope were at a distance of 24 feet from the slit.

[1] Parkinson's *Optics*, § 96. Cor. 2.

1049. Dark Lines in the Solar Spectrum.—When a pure spectrum of solar light is examined by any of these methods, it is seen to be traversed by numerous dark lines, constituting, if we may so say, dark images of the slit. Each of these is an indication that a particular kind of elementary ray is wanting[1] in solar light. Every elementary ray that is present gives its own image of the slit in its own peculiar colour; and these images are arranged in strict contiguity, so as to form a continuous band of light passing by perfectly gradual transitions through the whole range of simple colour, except at the narrow intervals occupied by the dark lines. Fig. 1, Plate III., is a rough representation of the appearance thus presented. If the slit is illuminated by a gas flame, or by any ordinary lamp, instead of by solar light, no such lines are seen, but a perfectly continuous spectrum is obtained. The dark lines are therefore not characteristic of light in general, but only of solar light.

Wollaston saw and described some of the more conspicuous of them. Fraunhofer counted about 600, and marked the places of 354 upon a map of the spectrum, distinguishing some of the more conspicuous by the names of letters of the alphabet, as indicated in fig. 1. These lines are constantly referred to as reference marks for the accurate specification of different portions of the spectrum. They always occur in precisely the same places as regards colour, but do not retain exactly the same relative distances one from another, when prisms of different materials are employed, different parts of the spectrum being unequally expanded by different refracting substances.[2] The inequality, however, is not so great as to introduce any difficulty in the identification of the lines.

The dark lines in the solar spectrum are often called Fraunhofer's lines. Fraunhofer himself called them the "fixed lines."

1050. Invisible Rays of the Spectrum.—The brightness of the solar spectrum, however obtained, is by no means equal throughout, but is greatest between the dark lines D and E; that is to say, in the yellow and the neighbouring colours orange and light green; and falls off gradually on both sides.

The heating effect upon a small thermometer or thermopile increases in going from the violet to the red, and still continues to increase for a certain distance beyond the visible spectrum at the red end. Prisms and lenses of rock-salt should be employed for this

[1] Probably not absolutely wanting, but so feeble as to appear black by contrast.

[2] This property is called the *irrationality of dispersion.*

investigation, as glass largely absorbs the invisible rays which lie beyond the red.

When the spectrum is thrown upon the sensitized paper employed in photography, the action is very feeble in the red, strong in the blue and violet, and is sensible to a great distance beyond the violet end. When proper precautions are taken to insure a very pure spectrum, the photograph reveals the existence of dark lines, like those of Fraunhofer, in the invisible ultra-violet portion of the spectrum. The strongest of these have been named L, M, N, O, P.

1051. **Phosphorescence and Fluorescence.**—There are some substances which, after being exposed in the sun, are found for a long time to appear self-luminous when viewed in the dark, and this

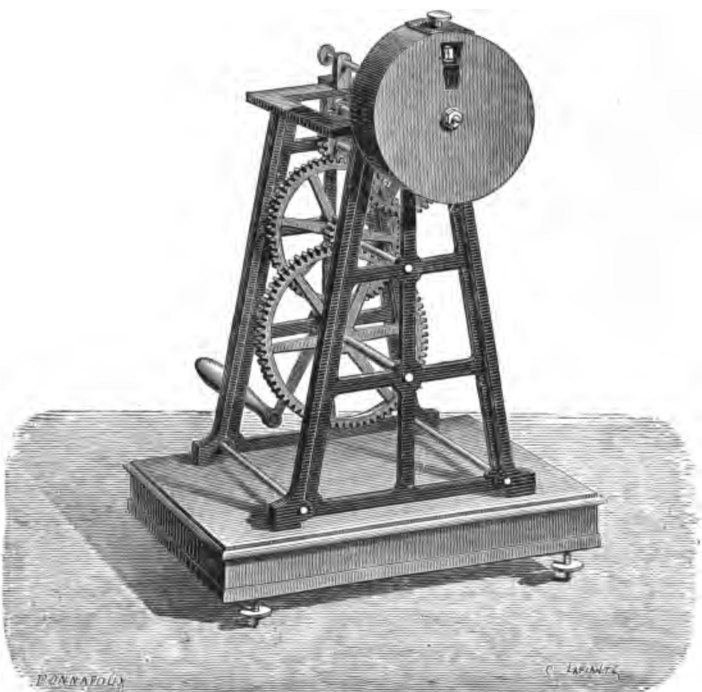


Fig. 758.—Becquerel's Phosphoroscope.

without any signs of combustion or sensible elevation of temperature. Such substances are called *phosphorescent.* Sulphuret of calcium and sulphuret of barium have long been noted for this property, and have hence been called respectively *Canton's phosphorus,* and *Bologna*

phosphorus. The phenomenon is chiefly due to the action of the violet and ultra-violet portion of the sun's rays.

More recent investigations have shown that the same property exists in a much lower degree in an immense number of bodies, their phosphorescence continuing, in most cases, only for a fraction of a second after their withdrawal from the sun's rays. E. Becquerel has contrived an instrument, called the *phosphoroscope,* which is extremely appropriate for the observation of this phenomenon. It is represented in Fig. 758. Its most characteristic feature is a pair of rigidly connected discs (Fig. 759), each pierced with four openings, those of the one being not opposite but midway between those of the other. This pair of discs can be set in very rapid rotation by means of a series of wheels and pinions. The body to be examined is attached to a fixed stand between the two discs, so that it is alternately exposed on opposite sides as the discs rotate. One side is turned towards the sun, and the other towards the observer, who accordingly only sees the body when it is not exposed to the sun's rays. The cylindrical case within which the discs revolve, is fitted into a hole in the shutter of a dark room, and is pierced with an opening on each side exactly opposite the position in which the body is fixed. The body, if not phosphorescent, will never be seen by the observer, as it is always in darkness except when it is hidden by the intervening disc. If its phosphorescence lasts as long as an eighth part of the time of one rotation, it will become visible in the darkness.

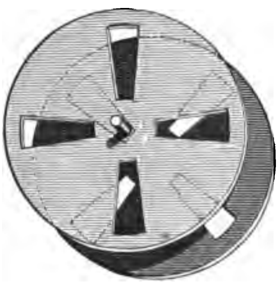


Fig. 759.
Discs of Phosphoroscope.

Nearly all bodies, when thus examined, show traces of phosphorescence, lasting, however, in some cases, only for a ten-thousandth of a second.

The phenomenon of *fluorescence,* which is illustrated in Plate II. accompanying § 817, appears to be essentially identical with phosphorescence. The former name is applied to the phenomenon, if it is observed while the body is actually exposed to the source of light, the latter to the effect of the same kind, but usually less intense, which is observed after the light from the source is cut off. Both forms of the phenomenon occur in a strongly-marked degree in the same bodies. Canary-glass, which is coloured with oxide of uranium, is

a very convenient material for the exhibition of fluorescence. A thick piece of it, held in the violet or ultra-violet portion of the solar spectrum, is filled to the depth of from $\frac{1}{8}$ to $\frac{1}{4}$ of an inch with a faint nebulous light. A solution of sulphate of quinine is also frequently employed for exhibiting the same effect, the luminosity in this case being bluish. If the solar spectrum be thrown upon a screen freshly washed with sulphate of quinine, the ultra-violet portion will become visible by fluorescence; and if the spectrum be very pure, the presence of dark lines in this portion will be detected.

The light of the electric lamp is particularly rich in ultra-violet rays, this portion of its spectrum being much longer than in the case of solar light, and about twice as long as the spectrum of luminous rays. Prisms and lenses of quartz should be employed for this purpose, as this material is specially transparent to the highly-refrangible rays. Flint-glass prisms, however, if of good quality, answer well in operating on solar light. The luminosity produced by fluorescence has sensibly the same tint in all parts of the spectrum in which it occurs, and depends upon the fluorescent substance employed. Prismatic analysis is not necessary to the exhibition of fluorescence. The phenomenon is very conspicuous when the electric discharge of a Holtz's machine or a Ruhmkorff's coil is passed near fluorescent substances, and it is faintly visible when these substances are examined in bright sunshine. The light emitted by a fluorescent substance is found by analysis not to be homogeneous, but to consist of rays having a wide range of refrangibility.

The ultra-violet rays, though usually styled invisible, are not altogether deserving of this title. By keeping all the rest of the spectrum out of sight, and carefully excluding all extraneous light, the eye is enabled to perceive these highly refrangible rays. Their colour is described as lavender-gray or bluish white, and has been attributed, with much appearance of probability, to fluorescence of the retina. The ultra-red rays, on the other hand, are never seen; but this may be owing to the fact, which has been established by experiment, that they are largely, if not entirely, absorbed before they can reach the retina.

1052. Recomposition of White Light.—The composite nature of white light can be established by actual synthesis. This can be done in several ways.

1. If a second prism, precisely similar to the first, but with its refracting edge turned the contrary way, is interposed in the path of

the coloured beam, very near its place of emergence from the first prism, the deviation produced by the second prism will be equal and opposite to that produced by the first, the two prisms will produce the effect of a parallel plate, and the image on the screen will be a white spot, nearly in the same position as if the prisms were removed.

2. Let a convex lens (Fig. 760) be interposed in the path of the coloured beam, in such a manner that it receives all the rays, and

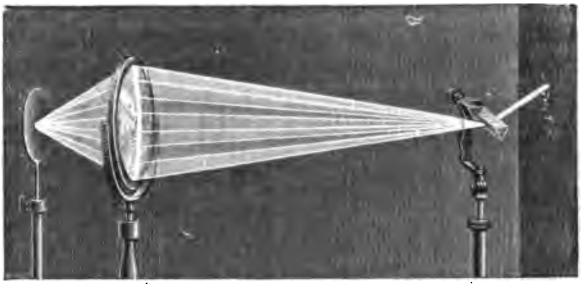


Fig. 760.—Recomposition by Lens.

that the screen and the prism are at conjugate focal distances. The image thus obtained on the screen will be white, at least in its central portions.

3. Let a number of plane mirrors be placed so as to receive the successive coloured rays, and to reflect them all to one point of a

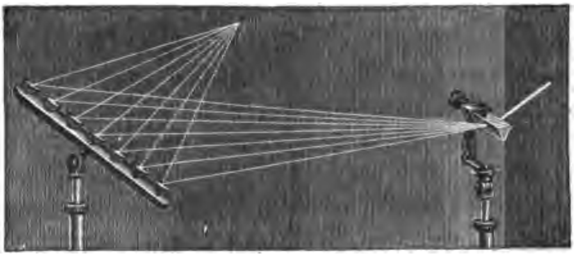


Fig. 761.—Recomposition by Mirrors.

screen, as in Fig. 761. The bright spot thus formed will be white or approximately white.

More complete information respecting the mixture of colours will be given in the next chapter.

1053. Spectroscope.—When we have obtained a pure spectrum by any of the methods above indicated, we have in fact effected an analysis of the light with which the slit is illuminated. In recent years, many forms of apparatus have been constructed for this purpose, under the name of *spectroscopes.*

A spectroscope usually contains, besides a slit, a prism, and a telescope (as in Fraunhofer's method of observation), a convex lens called a *collimator*, which is fixed between the prism and the slit, at the distance of its principal focal length from the latter. The effect of this arrangement is, that rays from any point of the slit emerge parallel, as if they came from a much larger slit (the virtual image of the real slit) at a much greater distance. The prism (set at minimum deviation) forms a virtual image of this image at the same distance, but in a different direction, on the principle of Fig. 757.

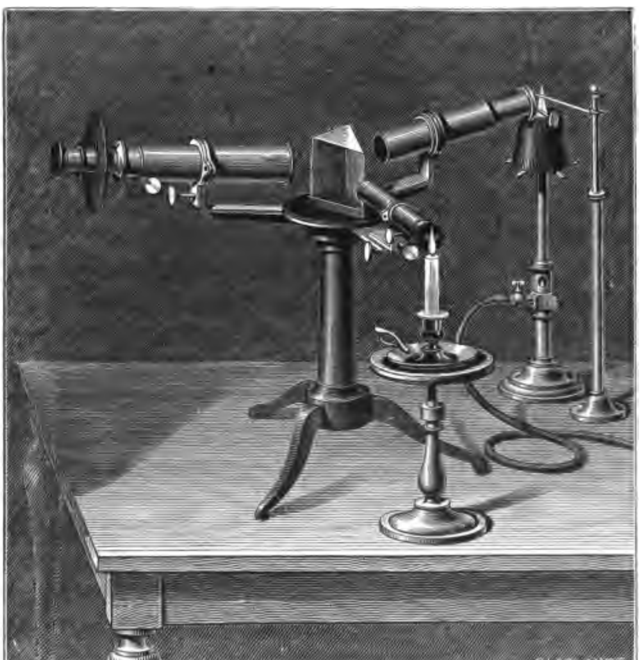


Fig. 762.—Spectroscope.

To this second virtual image the telescope is directed, being focussed as if for a very distant object.

Fig. 762 represents a spectroscope thus constructed. The tube of

the collimator is the further tube in the figure, the lens being at the end of the tube next the prism, while at the far end, close to the lamp flame, there is a slit (not visible in the figure) consisting of an opening between two parallel knife-edges, one of which can be moved to or from the other by turning a screw. The knife-edges must be very true, both as regards straightness and parallelism, as it is often necessary to make the slit exceedingly narrow. The tube on the left hand is the telescope, furnished with a broad guard to screen the eye from extraneous light. The near tube, with a candle opposite its end, is for purposes of measurement. It contains, at the end next the candle, a scale of equal parts, engraved or photographed on glass. At the other end of the tube is a collimating lens, at the distance of its own focal length from the scale; and the collimator is set so that its axis and the axis of the telescope make equal angles with the near face of the prism. The observer thus sees in the telescope, by reflection from the surface of the prism, a magnified image of the scale, serving as a standard of reference for assigning the positions of the lines in any spectrum which may be under examination. This arrangement affords great facilities for rapid observation.

Another plan is, for the arm which carries the telescope to be movable round a graduated circle, the telescope being furnished with cross-wires, which the observer must bring into coincidence with any line whose position he desires to measure.

Arrangements are frequently made for seeing the spectra of two different sources of light in the same field of view, one half of the length of the slit being illuminated by the direct rays of one of the sources, while a reflector, placed opposite the other half of the slit, supplies it with reflected light derived from the other source. This method should always be employed when there is a question as to the exact coincidence of lines in the two spectra. The reflector is usually an equilateral prism. The light enters normally at one of its faces, is totally reflected at another, and emerges normally at the third, as in the annexed sketch (Fig. 763), where the dotted line represents the path of a ray.

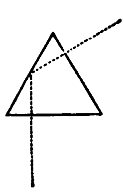


Fig. 763. Reflecting Prism.

A one-prism spectroscope is amply sufficient for the ordinary purposes of chemistry. For some astronomical applications a much greater dispersion is required. This is attained by making the light pass through a number of prisms in succession, each being set in the proper position for giving minimum deviation to the rays which have

passed through its predecessor. Fig. 764 represents the ground plan of such a battery of prisms, and shows the gradually increasing width of a pencil as it passes round the series of nine prisms on its way from the collimator to the telescope. The prisms are usually connected by a special arrangement, which enables the observer, by a single movement, to bring all the prisms at once into the proper position for giving minimum deviation to the particular ray under examination, a position which differs considerably for rays of different refrangibilities.

Fig. 764.—Train of Prisms.

1054. Use of Collimator.—The introduction of a collimating lens, to be used in conjunction with a prism and observing telescope, is due to Professor Swan.[1] Fraunhofer employed no collimator; but his prism was at a distance of 24 feet from the slit, whereas a distance of less than 1 foot suffices when a collimator is used.

It is obvious that homogeneous light, coming from a point at the distance of a foot, and falling upon the whole of one face of a prism —say an inch in width, cannot all have the incidence proper for minimum deviation. Those rays which very nearly fulfil this condition, will concur in forming a tolerably sharp image, in the position which we have already indicated. The emergent rays taken as a whole, do not diverge from any one point, but are tangents to a virtual caustic (§ 974). An eye receiving any portion of these rays, will see an image in the direction of a tangent from the eye to the caustic; and this image will be the more blurred as the deviation is further from the minimum. When the naked eye is employed, and the prism is so adjusted that the centre of the pupil receives rays of minimum deviation, a distance of five or six feet between the prism and slit is sufficient to give a sharp image; but if we employ an observing telescope whose object-glass is five times larger in diameter than the pupil of the eye, we must increase the distance between the

[1] *Trans. Roy. Soc. Edinburgh*, 1847 and 1856.

prism and slit fivefold to obtain equally good definition. A collimating lens, if achromatic and of good quality, gives the advantage of good definition without inconvenient length.

When exact measures of deviation are required, it confers the further advantage of altogether dispensing with a very troublesome correction for parallax.

1055. Different Kinds of Spectra.—The examination of a great variety of sources of light has shown that spectra may be divided into the following classes:—

1. The solar spectrum is characterized, as already observed, by a definite system of dark lines interrupting an otherwise continuous succession of colours. The same system of dark lines is found in the spectra of the moon and planets, this being merely a consequence of the fact that they shine by the reflected light of the sun. The spectra of the fixed stars also contain systems of dark lines, which are different for different stars.

2. The spectra of incandescent solids and liquids are completely continuous, containing light of all refrangibilities from the extreme red to a higher limit depending on the temperature.

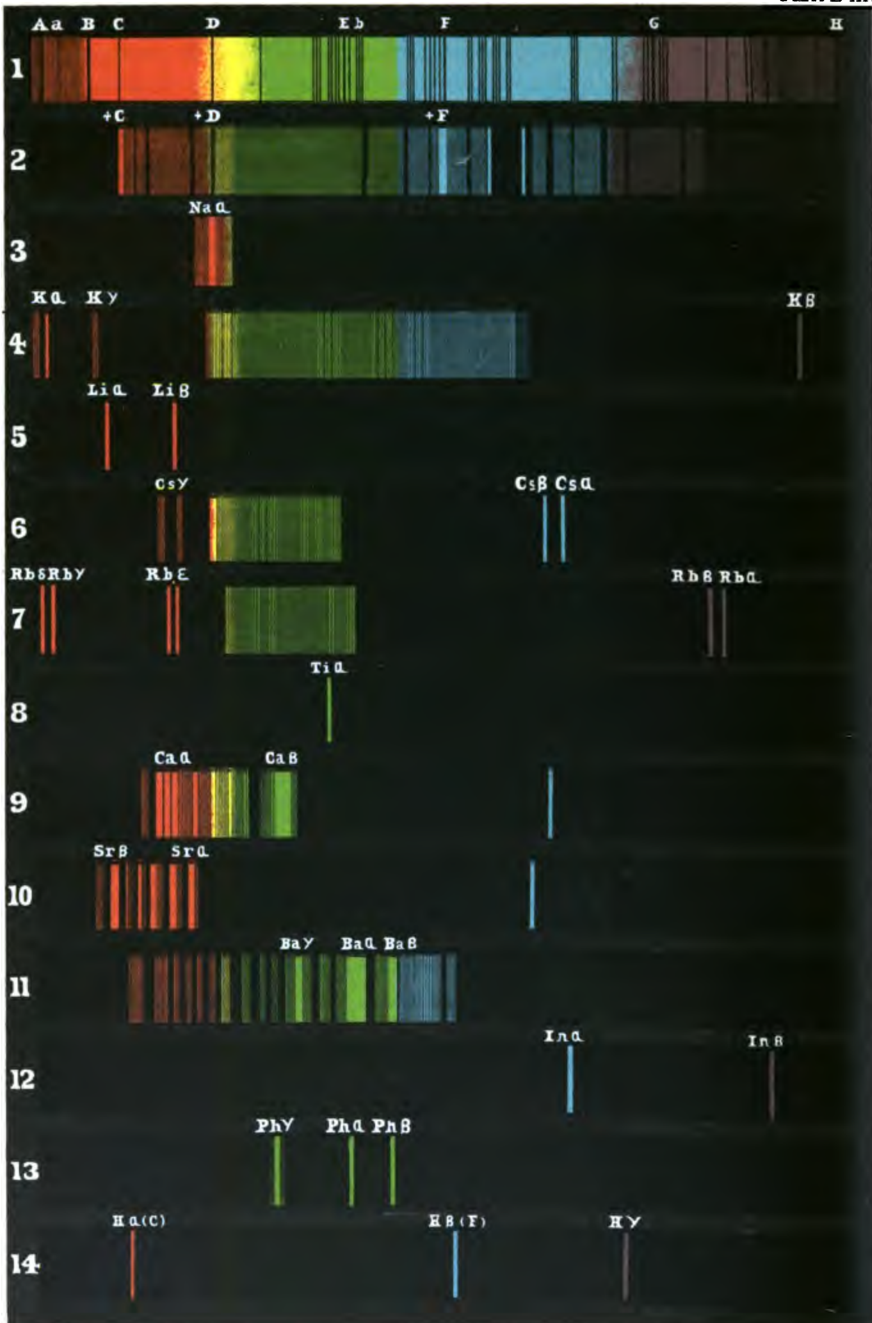
3. Flames not containing solid particles in suspension, but merely emitting the light of incandescent gases, give a discontinuous spectrum, consisting of a finite number of bright lines. The continuity of the spectrum of a gas or candle flame, arises from the fact that nearly all the light of the flame is emitted by incandescent particles of solid carbon,—particles which we can easily collect in the form of soot. When a gas-flame is fed with an excessive quantity of air, as in Bunsen's burner, the separation of the solid particles of carbon from the hydrogen with which they were combined, no longer takes place; the combustion is purely gaseous, and the spectrum of the flame is found to consist of bright lines. When the electric light is produced between metallic terminals, its spectrum contains bright lines due to the incandescent vapour of these metals, together with other bright lines due to the incandescence of the oxygen and nitrogen of the air. When it is taken between charcoal terminals, its spectrum is continuous; but if metallic particles be present, the bright lines due to their vapours can be seen as well.

The spectrum of the electric discharge in a Geissler's tube consists of bright lines characteristic of the gas contained in the tube.

1056. Spectrum Analysis.—As the spectrum exhibited by a compound substance when subjected to the action of heat, is frequently

SPECTRA OF VARIOUS SOURCES OF LIGHT.

PLATE III.



1. *The Sun.* 2. *The Sun's edge.* 3. *Sodium.* 4. *Potassium.* 5. *Lithium.* 6. *Caesium.* 7. *Rubidium.* 8. *Thallium.* 9. *Calcium.* 10. *Strontium.* 11. *Barium.* 12. *Indium.* 13. *Phosphorus.* 14. *Hydrogen.*

to be identical with the spe... of the spectra of its constit... is an exceedingly rea... analysis.

A salt of a metal which is easily ... lamp-flame, by means of a ... which form the spectrum of the ... spectroscope directed to the flame. ... flame itself is too faint to inte... metals which require a higher temperat... discharge is usually ... for gases.

Plate III. contains representati... of several of the easily volatilized metals, as ... hydrogen, and the solar spectrum is ... for comparison. ... bright lines of some of these substances ... exactly coincident ... some of the dark lines in the solar spectrum.

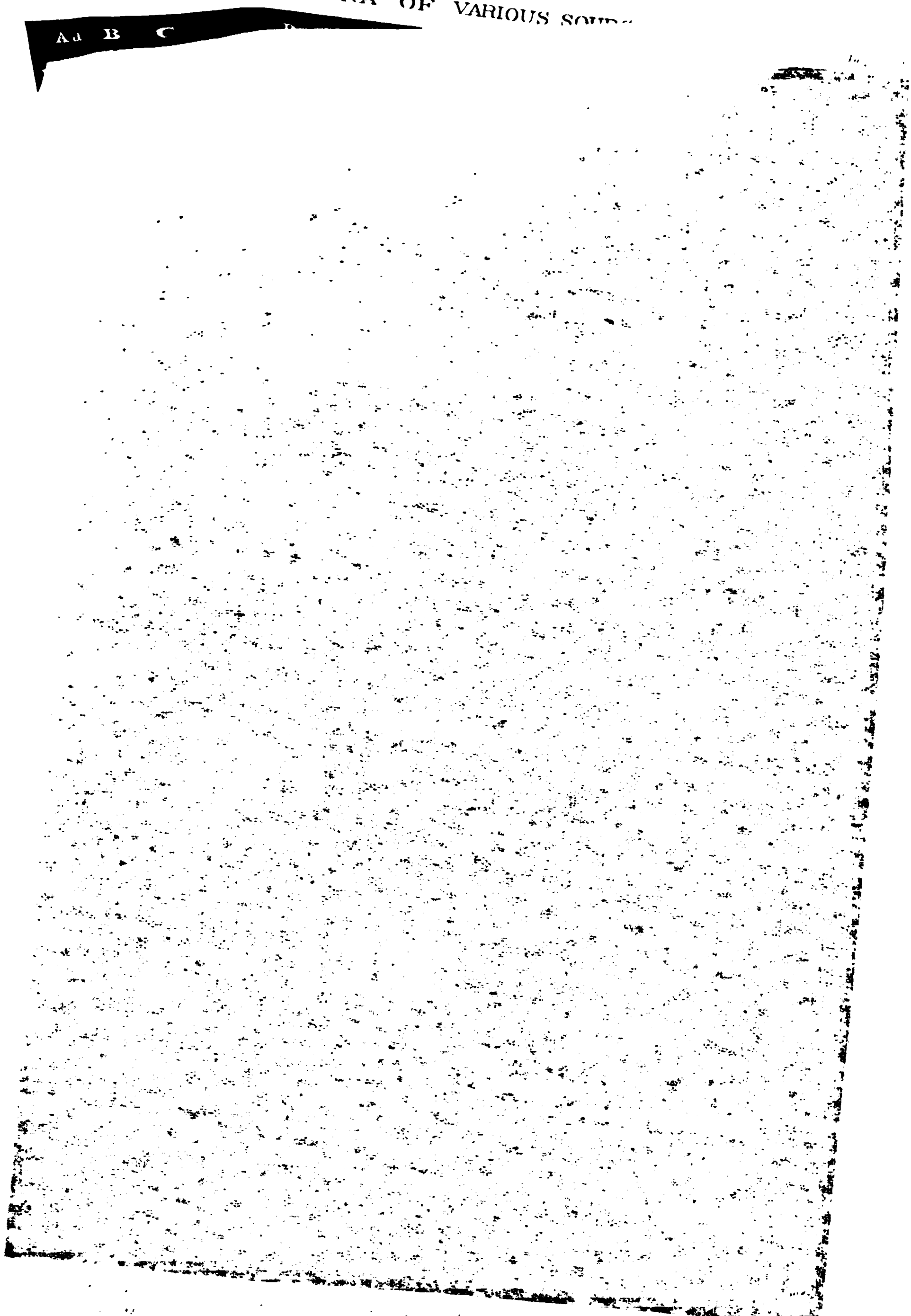
The fact that certain substances when ... definite ... lines, has been known for many years ... researches of Brewster, Herschel, Talbot, and ... long time thought that the same line might ... especially as the bright yellow line of ... was seen in flames in which ... was not supposed to be present. Professor Swan, having ascertained that the presence of 2,500,000th part of a grain ... sodium in a flame was sufficient ... produce it, considered himself justified in asserting, in 1856, that ... line was always to be taken as an indication of the presence of ... in larger or smaller quantity.

... the greatest advance in spectral a... was made by Bunsen ... Kirchhoff, who, by means of a four-... obtained accurate observations of the positions of the ... lines in the spectra of a great number of substances, as well as ... dark lines ... the solar spectrum, and called attention to the ... the latter with several of the former. Since the ... their researches the spectroscope has come into general ... chemists, and has already led to the discovery of four ... cæsium, rubidium, thallium, and indium.

257. Reversal of Bright Lines. Analysis of the Sun's Atmosphere.

These appear to be merely examples of the dissociation of the elements of a chemical ... found at high temperatures.

SPECTRA OF VARIOUS SOUR…



found to be identical with the spectrum of one of its constituents, or to consist of the spectra of its constituents superimposed,[1] the spectroscope affords an exceedingly ready method of performing qualitative analysis.

If a salt of a metal which is easily volatilized is introduced into a Bunsen lamp-flame, by means of a loop of platinum wire, the bright lines which form the spectrum of the metal will at once be seen in a spectroscope directed to the flame; and the spectrum of the Bunsen flame itself is too faint to introduce any confusion. For those metals which require a higher temperature to volatilize them, electric discharge is usually employed. Geissler's tubes are commonly used for gases.

Plate III. contains representations of the spectra of several of the more easily volatilized metals, as well as of phosphorus and hydrogen; and the solar spectrum is given at the top for comparison. The bright lines of some of these substances are precisely coincident with some of the dark lines in the solar spectrum.

The fact that certain substances when incandescent give definite bright lines, has been known for many years, from the researches of Brewster, Herschel, Talbot, and others; but it was for a long time thought that the same line might be produced by different substances, more especially as the bright yellow line of sodium was often seen in flames in which that metal was not supposed to be present. Professor Swan, having ascertained that the presence of the 2,500,000th part of a grain of sodium in a flame was sufficient to produce it, considered himself justified in asserting, in 1856, that this line was always to be taken as an indication of the presence of sodium in larger or smaller quantity.

But the greatest advance in spectral analysis was made by Bunsen and Kirchhoff, who, by means of a four-prism spectroscope, obtained accurate observations of the positions of the bright lines in the spectra of a great number of substances, as well as of the dark lines in the solar spectrum, and called attention to the identity of several of the latter with several of the former. Since the publication of their researches, the spectroscope has come into general use among chemists, and has already led to the discovery of four new metals, cæsium, rubidium, thallium, and indium.

1057. Reversal of Bright Lines. Analysis of the Sun's Atmosphere.

[1] These appear to be merely examples of the dissociation of the elements of a chemical compound at high temperatures.

—It may seem surprising that, while incandescent solids and liquids are found to give continuous spectra containing rays of all refrangibilities, the solar spectrum is interrupted by dark lines indicating the absence or relative feebleness of certain elementary rays. It seems natural to suppose that the deficient rays have been removed by selective absorption, and this conjecture was thrown out long since. But where and how is this absorption produced? These questions have now received an answer which appears completely satisfactory.

According to the theory of exchanges, which has been explained in connection with the radiation of heat (§ 464, 483), every substance which emits certain kinds of rays to the exclusion of others, absorbs the same kind which it emits; and when its temperature is the same in the two cases compared, its emissive and absorbing power are precisely equal for any one elementary ray.

When an incandescent vapour, emitting only rays of certain definite refrangibilities, and therefore having a spectrum of bright lines, is interposed between the observer and a very bright source of light, giving a continuous spectrum, the vapour allows no rays of its own peculiar kinds to pass; so that the light which actually comes to the observer consists of transmitted rays in which these particular kinds are wanting, together with the rays emitted by the vapour itself, these latter being of precisely the same kind as those which it has refused to transmit. It depends on the relative brightness of the two sources whether these particular rays shall be on the whole in excess or defect as compared with the rest. If the two sources are at all comparable in brightness, these rays will be greatly in excess, inasmuch as they constitute the whole light of the one, and only a minute fraction of the light of the other; but the light of the electric lamp, or of the lime-light, is usually found sufficiently powerful to produce the contrary effect; so that if, for example, a spirit-lamp with salted wick is interposed between the slit of a spectroscope and the electric light, the bright yellow line due to the sodium appears black by contrast with the much brighter back-ground which belongs to the continuous spectrum of the charcoal points. By employing only some 10 or 15 cells, a light may be obtained, the yellow portion of which, as seen in a one-prism spectroscope, is sensibly equal in brightness to the yellow line of the sodium flame, so that this line can no longer be separately detected, and the appearance is the same whether the sodium flame be interposed or removed.

The dark lines in the solar spectrum would therefore be accounted

for by supposing that the principal portion of the sun's light comes from an inner stratum which gives a continuous spectrum, and that a layer external to this contains vapours which absorb particular rays, and thus produce the dark lines. The stratum which gives the continuous spectrum might be solid, liquid, or even gaseous, for the experiments of Frankland and Lockyer have shown that, as the pressure of a gas is increased, its bright lines broaden out into bands, and that the bands at length become so wide as to join each other and form a continuous spectrum.[1]

Hydrogen, sodium, calcium, barium, magnesium, zinc, iron, chromium, cobalt, nickel, copper, and manganese have all been proved to exist in the sun by the accurate identity of position of their bright lines with certain dark lines in the sun's spectrum.

The strong line D, which in a good instrument is seen to consist of two lines near together, is due to sodium; and the lines C and F are due to hydrogen. No less than 450 of the solar dark lines have been identified with bright lines of iron.

1058. Telespectroscope. Solar Prominences.—For astronomical investigations, the spectroscope is usually fitted to a telescope, and takes the place of the eye-piece, the plane of the slit being placed in the principal focus of the object-glass, so that the image is thrown upon it, and the light which enters the slit is the light which forms one strip (so to speak) of the image, and which therefore comes from one strip of the object. A telescope thus equipped is called a telespectroscope. Extremely interesting results have been obtained by thus subjecting to examination a strip of the sun's edge, the strip being sometimes tangential to the sun's disc, and sometimes radial. When the former arrangement is adopted, the appearance presented is that depicted in No. 2, Plate III., consisting of a few bright lines scattered through a back-ground of the ordinary solar spectrum. The bright lines are due to an outer layer called the *chromosphere*, which is thus proved to be vaporous. The ordinary solar spectrum which accompanies it, is due to that part of the sun from which most of our light is derived. This part is called the *photosphere*, and if not solid or liquid, it must consist of vapour so highly compressed that its properties approximate to those of a liquid.

When the slit is placed radially, in such a position that only a

[1] The gradual transition from a spectrum of bright lines to a continuous spectrum may be held to be an illustration of the continuous transition which can be effected from the condition of ordinary gas to that of ordinary liquid (§ 380).

small portion of its length receives light from the body of the sun, the spectra of the photosphere and chromosphere are seen in immediate contiguity, and the bright lines in the latter (notably those of hydrogen, No. 14, Plate III.) are observed to form continuations of some of the dark lines of the former.

The chromosphere is so much less bright than the photosphere, that, until a few years since, its existence was never revealed except during total eclipses of the sun, when projecting portions of it were seen extending beyond the dark body of the moon. The spectrum of these projecting portions, which have been variously called "prominences," "red flames," and "rose-coloured protuberances," was first observed during the "Indian eclipse" of 1868, and was found to consist of bright lines, including those of hydrogen. From their excessive brightness, M. Janssen, who was one of the observers, expressed confidence that he should be able to see them in full sunshine; and the same idea had been already conceived and published by Mr. Lockyer. The expectation was shortly afterwards realized by both these observers, and the chromosphere has ever since been an object of frequent observation. The visibility of the chromosphere lines in full sunshine, depends upon the principle that, while a continuous spectrum is extended, and therefore made fainter, by increased dispersion, a bright line in a spectrum is not sensibly broadened, and therefore loses very little of its intrinsic brightness (§ 1061). Very high dispersion is necessary for this purpose.

Still more recently, by opening the slit to about the average width of the prominence-region, as measured on the image of the sun which is thrown on the slit, it has been found possible to see the whole of an average-sized prominence at one view. This will be understood by remembering that a bright line as seen in a spectrum is a monochromatic image of the illuminated portion of the slit, or when a telespectroscope is used, as in the present case, it is a monochromatic image of one strip of the image formed by the object-glass, namely, that strip which coincides with the slit. If this strip then contains a prominence in which the elementary rays C and F (No. 2, Plate III.) are much stronger than in the rest of the strip, a red image of the prominence will be seen in the part of the spectrum corresponding to the line C, and a blue image in the place corresponding to the line F. This method of observation requires greater dispersion than is necessary for the mere detection of the chromosphere lines; the dispersion required for enabling a bright-line spectrum to predomi-

nate over a continuous spectrum being always nearly proportional to the width of the slit (§ 1061).

Of the nebulæ, it is well known that some have been resolved by powerful telescopes into clusters of stars, while others have as yet proved irresolvable. Huggins has found that the former class of nebulæ give spectra of the same general character as the sun and the fixed stars, but that some of the latter class give spectra of bright lines, indicating that their constitution is gaseous.

1059. Displacement of Lines consequent on Celestial Motions.—According to the undulatory theory of light, which is now universally accepted, the fundamental difference between the different rays which compose the complete spectrum, is a difference of wave-frequency, and, as connected with this, a difference of wave-length in any given medium, the rays of greatest wave-frequency or shortest wave-length being the most refrangible.

Doppler first called attention to the change of refrangibility which must be expected to ensue from the mutual approach or recess of the observer and the source of light, the expectation being grounded on reasoning which we have explained in connection with acoustics (§ 898).

Doppler adduced this principle to explain the colours of the fixed stars, a purpose to which it is quite inadequate; but it has rendered very important service in connection with spectroscopic research. Displacement of a line towards the more refrangible end of the spectrum, indicates approach, displacement in the opposite direction indicates recess, and the velocity of approach or recess admits of being calculated from the observed displacement.

When the slit of the spectroscope crosses a spot on the sun's disc, the dark lines lose their straightness in this part, and are bent, sometimes to one side, sometimes to the other. These appearances clearly indicate uprush and downrush of gases in the sun's atmosphere in the region occupied by the spot.

Huggins detected a displacement of the F line towards the red end, in the spectrum of Sirius, as compared with the spectrum of the sun or of hydrogen. The displacement is so small as only to admit of measurement by very powerful instrumental appliances; but, small as it is, calculation shows that it indicates a motion of recess at the rate of about 30 miles per second.[1]

[1] The observed displacement corresponded to recess at the rate of 41·4 miles per second; but 12·0 of this must be deducted for the motion of the earth in its orbit at the season of

1060. Spectra of Artificial Lights.—The spectra of the artificial lights in ordinary use (including gas, oil-lamps, and candles) differ from the solar spectrum in the relative brightness of the different colours, as well as in the entire absence of dark lines. They are comparatively strong in red and green, but weak in blue; hence all colours which contain much blue in their composition appear to disadvantage by gas-light.

It is possible to find artificial lights whose spectra are of a completely different character. The salts of strontium, for example, give red light, composed of the ingredients represented in spectrum No. 10, Plate III., and those of sodium yellow light (No. 3, Plate III.). If a room is illuminated by a sodium flame (for example, by a spirit-lamp with salt sprinkled on the wick), all objects in the room will appear of a uniform colour (that of the flame itself), differing only in brightness, those which contain no yellow in their spectrum as seen by day-light being changed to black. The human countenance and hands assume a ghastly hue, and the lips are no longer red.

A similar phenomenon is observed when a coloured body is held in different parts of the solar spectrum in a dark room, so as to be illuminated by different kinds of monochromatic light. The object either appears of the same colour as the light which falls upon it, or else it refuses to reflect this light and appears black. Hence a screen for exhibiting the spectrum should be white.

1061. Brightness and Purity.—The laws which determine the brightness of images generally, and which have been expounded at some length in the preceding chapter, may be applied to the spectroscope. We shall, in the first instance, neglect the loss of light by reflection and imperfect transmission.

Let Δ denote the *prismatic dispersion*, as measured by the angular separation of two specified monochromatic images when the naked eye is applied to the last prism, the observing telescope being removed. Then, putting m for the linear magnifying power of the

the year when the observation was made. The remainder, 29·4, was therefore the rate at which the distance between the sun and Sirius was increasing.

In a more recent paper Dr. Huggins gave the results of observations with more powerful instrumental appliances. The recess of Sirius was found to be only 20 miles per second. Arcturus was found to be approaching at the rate of 50 miles per second. Community of motion was established in certain sets of stars; and the belief previously held by astronomers, as to the direction in which the solar system is moving with respect to the stars as a whole, was fully confirmed.

telescope, $m\Delta$ is the angular separation observed when the eye is applied to the telescope. We shall call $m\Delta$ the *total dispersion*.

Let θ denote the angle which the breadth of the slit subtends at the centre of the collimating lens, and which is measured by $\frac{\text{breadth of slit}}{\text{focal length of lens}}$. Then θ is also the apparent breadth of any absolutely monochromatic image of the slit, formed by rays of minimum deviation, as seen by an eye applied either to the first prism, the last prism, or any one of the train of prisms. The change produced in a pencil of monochromatic rays by transmission through a prism at minimum deviation, is in fact simply a change of direction, without any change of mutual inclination; and thus neither brightness nor apparent size is at all affected. In ordinary cases, the bright lines of a spectrum may be regarded as monochromatic, and their apparent breadth, as seen without the telescope, is sensibly equal to θ. Strictly speaking, the effect of prismatic dispersion in actual cases, is to increase the apparent breadth by a small quantity, which, if all the prisms are alike, is proportional to the number of prisms; but the increase is usually too small to be sensible.

Let I denote the intrinsic brightness of the source as regards any one of its (approximately) monochromatic constituents; in other words, the brightness which the source would have if deprived of all its light except that which goes to form a particular bright line. Then, still neglecting the light stopped by the instrument, the brightness of this line as seen without the aid of the telescope will be I; and as seen in the telescope it will either be equal to or less than this, according to the magnifying power of the telescope and the effective aperture of the object-glass (§ 1038). If the breadth of the slit be halved, the breadth of the bright line will be halved, and its brightness will be unchanged. These conclusions remain true so long as the bright line can be regarded as practically monochromatic.

The brightness of any part of a *continuous* spectrum follows a very different law. It varies directly as the width of the slit, and inversely as the prismatic dispersion. Its value without the observing telescope, or its maximum value with a telescope, is $\frac{\theta}{\Delta}i$, where i is a coefficient depending only on the source.

The *purity* of any part of a continuous spectrum is properly measured by the ratio of the *distance between two specified monochromatic images* to the *breadth of either*, the distance in question being measured from the centre of one to the centre of the other.

This ratio is unaffected by the employment of an observing telescope, and is $\frac{\Delta}{\theta}$.

The ratio of the brightness of a bright line to that of the adjacent portion of a continuous spectrum forming its back-ground, is $\frac{\Delta I}{\theta i}$, assuming the line to be so nearly monochromatic that the increase of its breadth produced by the dispersion of the prisms is an insignificant fraction of its whole breadth. As we widen the slit, and so increase θ, we must increase Δ in the same ratio, if we wish to preserve the same ratio of brightness. As $\frac{\Delta}{\theta}$ is increased indefinitely, the predominance of the bright lines does not increase indefinitely, but tends to a definite limit, namely, to the predominance which they would have in a perfectly pure spectrum of the given source.

The loss of light by reflection and imperfect transmission, increases with the number of surfaces of glass which are to be traversed; so that, with a long train of prisms and an observing telescope, the actual brightness will always be much less than the theoretical brightness as above computed.

The actual purity is always less than the theoretical purity, being greatly dependent on freedom from optical imperfections; and these can be much more completely avoided in lenses than in prisms. It is said that a single good prism, with a first-class collimator and telescope (as originally employed by Swan), gives a spectrum much more free from blurring than the modern multiprism spectroscopes, when the total dispersion $m\,\Delta$ is the same in both the cases compared.

1062. Chromatic Aberration.—The unequal refrangibility of the different elementary rays is a source of grave inconvenience in connection with lenses. The focal length of a lens depends upon its index of refraction, which of course increases with refrangibility, the focal length being shortest for the most refrangible rays. Thus a lens of uniform material will not form a single white image of a white object, but a series of images, of all the colours of the spectrum, arranged at different distances, the violet images being nearest, and the red most remote. If we place a screen anywhere in the series of images, it can only be in the right position for one colour. Every other colour will give a blurred image, and the superposition of them all produces the image actually formed on the screen. If the object be a uniform white spot on a black ground, its image on the screen

will consist of white in its central parts, gradually merging into a coloured fringe at its edge. Sharpness of outline is thus rendered impossible, and nothing better can be done than to place the screen at the focal distance corresponding to the brightest part of the spectrum. Similar indistinctness will attach to images observed in mid-air, whether directly or by means of another lens. This source of confusion is called *chromatic aberration.*

1063. Possibility of Achromatism.—In order to ascertain whether it was possible to remedy this evil by combining lenses of two different materials, Newton made some trials with a compound prism composed of glass and water (the latter containing a little sugar of lead), and he found that it was not possible, by any arrangement of these two substances, to produce deviation of the transmitted light without separation into its component colours. Unfortunately he did not extend his trials to other substances, but concluded at once that an *achromatic* prism (and hence also an achromatic lens) was an impossibility; and this conclusion was for a long time accepted as indisputable. Mr. Hall, a gentleman of Worcestershire, was the first to show that it was erroneous, and is said to have constructed some achromatic telescopes; but the important fact thus discovered did not become generally known till it was rediscovered by Dollond, an eminent London optician, in whose hands the manufacture of achromatic instruments attained great perfection.

1064. Conditions of Achromatism.—The conditions necessary for achromatism are easily explained. The angular separation between the brightest red and the brightest violet ray transmitted through a prism is called the *dispersion* of the prism, and is evidently the difference of the deviations of these rays. These deviations, for the position of minimum deviation of a prism of small refracting angle A, are $(\mu' - 1)$ A and $(\mu'' - 1)$ A, μ' and μ'' denoting the indices of refraction for the two rays considered (§ 1004, equation (1)) and their difference is $(\mu'' - \mu')$ A. This difference is always small in comparison with either of the deviations whose difference it is, and its ratio to either of them, or more accurately its ratio to the value of $(\mu - 1)$ A for the brightest part of the spectrum, is called the *dispersive power* of the substance. As the common factor A may be omitted, the formula for the dispersive power is evidently $\frac{\mu'' - \mu'}{\mu - 1}$.

If this ratio were the same for all substances, as Newton supposed, achromatism would be impossible; but in fact its value varies greatly,

and is greater for flint than for crown glass. If two prisms of these substances, of small refracting angles, be combined into one, with their edges turned opposite ways, they will achromatize one another if $(\mu'' - \mu')$ A, or the product of deviation by dispersive power, is the same for both. As the deviations can be made to have any ratio we please by altering the angles of the prisms, the condition is evidently possible.

The deviation which a simple ray undergoes in traversing a lens, at a distance x from the axis, is $\frac{x}{f}$, f denoting the focal length of the lens (§ 1004), and the separation of the red and violet constituents of a compound ray is the product of this deviation by the dispersive power of the material. If a convex and concave lens are combined, fitting closely together, the deviations which they produce in a ray traversing both, are in opposite directions, and so also are the dispersions. If we may regard x as having the same value for both (a supposition which amounts to neglecting the thicknesses of the lenses in comparison with their focal lengths) the condition of no resultant dispersion is that

$$\text{dispersive power} \times \frac{1}{f}$$

has the same value for both lenses. *Their focal lengths* must therefore be *directly as the dispersive powers of their materials.* These latter are about ·033 for crown and ·052 for flint glass. A converging achromatic lens usually consists of a double convex lens of crown fitted to a diverging meniscus of flint. In every achromatic combination of two pieces, the direction of resultant *deviation* is that due to the piece of smaller dispersive power.

The definition above given of dispersive power is rather loose. To make it accurate, we must specify, by reference to the "fixed lines," the precise positions of the two rays whose separation we consider.

Since the distances between the fixed lines have different proportions for crown and flint glass, achromatism of the whole spectrum is impossible. With two pieces it is possible to unite any two selected rays, with three pieces any three selected rays, and so on. It is considered a sign of good achromatism when no colours can be brought into view by bad focussing except purple and green.

1065. Achromatic Eye-pieces.—The eye-pieces of microscopes and astronomical telescopes, usually consist of two lenses of the same kind of glass, so arranged as to counteract, to some extent, the spherical

and chromatic aberrations of the object-glass. The *positive* eye-piece, invented by Ramsden, is suited for observation with cross-wires or micrometers; the *negative* eye-piece, invented by Huygens, is not adapted for purposes of measurement, but is preferred when distinct vision is the sole requisite. These eye-pieces are commonly called achromatic, but their achromatism is in a manner spurious. It consists not in bringing the red and violet images into true coincidence, but merely in causing one to cover the other as seen from the position occupied by the observer's eye.

In the best opera-glasses (§ 1033), the eye-piece, as well as the object-glass, is composed of lenses of flint and crown so combined as to be achromatic in the more proper sense of the word.

1066. Rainbow.—The unequal refrangibility of the different elementary rays furnishes a complete explanation of the ordinary phenomena of rainbows. The explanation was first given by Newton, who confirmed it by actual measurement.

It is well known that rainbows are seen when the sun is shining on drops of water. Sometimes one bow is seen, sometimes two, each of them presenting colours resembling those of the solar spectrum. When there is only one bow, the red arch is above and the violet below. When there is a second bow, it is at some distance outside of this, has the colours in reverse order, and is usually less bright.

Rainbows are often observed in the spray of cascades and fountains, when the sun is shining.

In every case, a line joining the observer to the sun is the axis of the bow or bows; that is to say, all parts of the length of the bow are at the same angular distance from the sun.

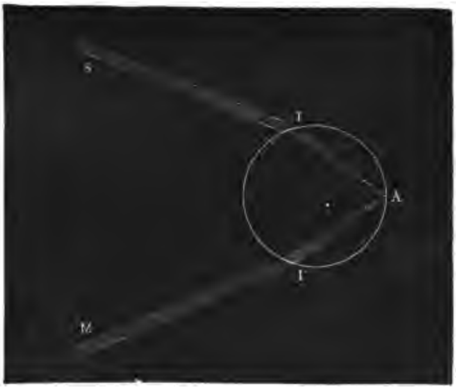


Fig. 765.—Production of Primary Bow.

The formation of the primary bow is illustrated by Fig. 765. A ray of solar light, falling on a spherical drop of water, in the direction S I, is refracted at I, then reflected internally from the back of the drop, and again refracted into the air in the

direction I′M. If we take different points of incidence, we shall obtain different directions of emergence, so that the whole light which emerges from the drop after undergoing, as in the figure, two refractions and one reflection, forms a widely-divergent pencil. Some portions of this pencil, however, contain very little light. This is especially the case with those rays which, having been incident nearly normally, are returned almost directly back, and also with those which were almost tangential at incidence. The greatest condensation, as regards any particular species of elementary ray, occurs at that part of the emergent pencil which has undergone *minimum deviation*. It is by means of rays which have undergone this minimum deviation, that the observer sees the corresponding colour in the bow; and the deviation which they have undergone is evidently equal to the angular distance of this part of the bow from the sun.

The minimum deviation will be greatest for those rays which are most refrangible. If the figure, for example, be supposed to represent the circumstances of minimum deviation for violet, we shall obtain smaller deviation in the case of red, even by giving the angle I A I′ the same value which it has in the case of minimum deviation for violet, and still more when we give it the value which corresponds to the minimum deviation of red. The most refrangible colours are accordingly seen furthest from the sun. The effect of the rays which undergo other than minimum deviation, is to produce a border of white light on the side remote from the sun; that is to say, on the inner edge of the bow.[1]

[1] When the drops are very uniform in size, a series of faint *supernumerary bows*, alternately purple and green, is sometimes seen beneath the primary bow. These bows are produced by the mutual interference of rays which have undergone other than minimum deviation, and the interference arises in the following way. Any two parallel directions of emergence, for rays of a given refrangibility, correspond in general to two different points of incidence on any given drop, one of the two incident rays being more nearly normal, and the other more nearly tangential to the drop than the ray of minimum deviation. These two rays have pursued dissimilar paths in the drop, and are in different phases when they reach the observer's eye. The difference of phase may amount to one, two, three, or more exact wave-lengths, and thus one, two, three, or more supernumerary bows may be formed. The distances between the supernumerary bows will be greater as the drops of water are smaller. This explanation is due to Dr. Thomas Young.

A more complete theory, in which diffraction is taken into account, is given by Airy in the *Cambridge Transactions* for 1838; and the volume for the following year contains an experimental verification by Miller. It appears from this theory that the maximum of intensity is less sharply marked than the ordinary theory would indicate, and does not correspond to the geometrical minimum of deviation, but to a deviation sensibly greater. Also that the region of sensible illumination extends beyond this geometrical minimum and shades off gradually.

The condensation which accompanies minimum deviation, is merely a particular case of the general mathematical law that magnitudes remain nearly constant in the neighbourhood of a maximum or minimum value. The rays which compose a small parallel pencil SI incident at and around the precise point which corresponds to minimum deviation, will thus have deviations which may be regarded as equal, and will accordingly remain sensibly parallel at emergence. A parallel pencil incident on any other part of the drop, will be divergent at emergence.

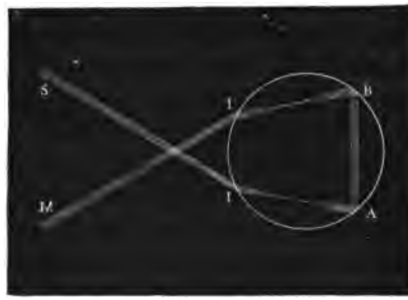


Fig. 766.—Production of Secondary Bow.

The indices of refraction for red and violet rays from air into water are respectively $\frac{108}{81}$ and $\frac{109}{81}$, and calculation shows that the distances from the centre of the sun to the parts of the bow in which these colours are strongest should be the supplements of 42° 2′ and 40° 17′ respectively. These results agree with observation. The angles 42° 2′ and 40° 17′ are the distances from the *anti-solar point*, which is always the centre of the bow.

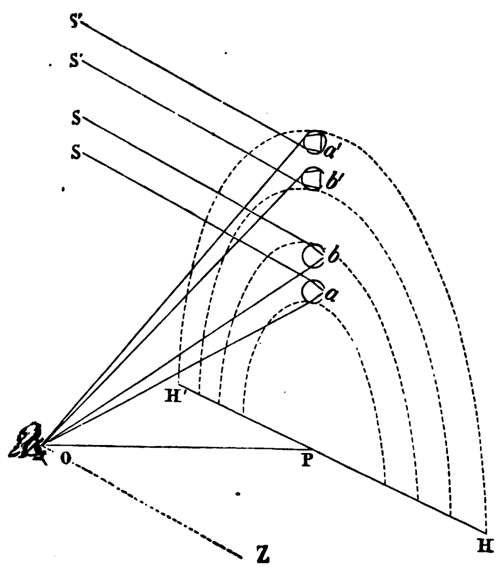


Fig. 767.—Relative Positions.

The rays which form the secondary bow have undergone two internal reflections, as represented in Fig. 766, and here again a special concentration occurs in the direction of minimum deviation. This deviation is greater than 180° and is greatest for the most refrangible rays. The distance of the arc thus formed from the sun's centre, is 360° *minus* the deviation, and is accord-

east for the most refrangible rays. Thus the violet arc est the sun, and the red furthest from it, in the secondary

Some idea of the relative situations of the eye, the sun, and the drops of water in which the two bows are formed, may be obtained from an inspection of Fig. 767.

SUNDRY ADDITIONS TO PREVIOUS CHAPTERS.

1066A. Goniometers.—A goniometer is an instrument for measuring the angle between two plane faces either of a crystal or of a prism. The measurement is usually made by means of reflections from the two faces. This may be done in either of the two following ways. For convenience of description we shall assume that the edge in which the two faces meet is vertical; in practice it may have any direction.

First method.—Observe in one of the two faces the reflection of an object at a few yards' distance, in the same horizontal plane with the prism; and by rotating the prism in this plane bring the image into apparent coincidence with some other object; or, if preferred, bring it upon the cross-wires of a fixed telescope. Then rotate the prism in the horizonal plane till the other face gives an image of the same object in the same position. The second face is now in or parallel to the position previously occupied by the first face, and the angle through which the prism has been turned is the angle between one face and the other face produced. By subtracting it from 180° we obtain the required angle between the faces. The goniometer is furnished with a graduated circle on which the rotation is read off.

Second method.—The goniometer must have a telescope (with cross-wires) which can travel round the graduated circle, while always directed towards its centre, where the prism stands. The prism is placed in such a position that rays from a distant object, or more conveniently from a slit in the focus of a collimating lens, fall upon both faces simultaneously. The telescope is first placed so as to receive on its cross-wires the image formed by reflection at one face, and is then moved past the base of the prism till it receives the image formed by reflection at the other face. The angle through

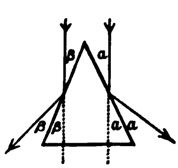


Fig. 767A.—Measurement of Angle of Prism.

which it has been moved is double the angle of the prism, as is obvious from Fig. 767A, in which the directions of the incident and reflected rays are represented by lines marked with arrowheads. The incident rays being parallel, $\alpha+\beta$ is the angle of the prism, and $2\alpha+2\beta$ is the angle between the reflected rays.

After measuring the angle of the prism by means of the observing telescope and collimator with slit, the index of refraction of the prism can be determined by illuminating the slit with sodium light or some other monochromatic light, and observing with the telescope the minimum deviation of the image of the slit formed by refraction through the prism. The index of refraction for the particular light employed can then be deduced by formula (4) page 1007.

1066B. Relation between Convex and Concave Mirrors.—The student should notice that every diagram relating to reflection from a concave mirror is equally applicable to a convex mirror. For example Fig. 683, page 988, correctly represents the virtual image AB of a real object *ba* in front of a convex mirror having C for its centre of curvature and F for its principal focus; and Fig. 677, page 982, shows the effect of interposing a convex mirror in the path of rays which are on their way to form a real image *ab*. The effect is to produce the virtual image AB. Or we may take AB as representing the real image which the rays were on their way to form, and then *ab* will be the virtual image which is formed instead. If the real image falls between the principal focus F and the convex mirror, the effect will be that instead of an inverted virtual image, an erect real image will be formed. Thus in Fig. 683, if AB be the image which the rays were on their way to form, the real image *ba* will be formed instead. Every diagram of an object and its image, as formed by a spherical mirror, has in fact four different interpretations, since the object and image may be interchanged, and the spherical surface may be polished on either side.

1066C. Nodal Points of a lens or system of lenses. When the thickness of a lens is considerable in comparison with the radii of curvature of its faces, the approximate assumption made in the first paragraph of page 1016, "that rays which pass through the centre of a lens undergo no deviation" is no longer admissible.

Referring to Fig. 718, page 1015, if we suppose the incident and emergent rays SI and RE to be produced to meet the axis in points N_1 and N_2, the ultimate positions of N_1 N_2 when the rays make very small angles with the axis are called the *nodal points* of the lens.

They are the two images of the centre of the lens formed by refraction out of the lens into air at the two surfaces. *Whenever the incident ray passes through the first nodal point, the emergent ray passes through the second nodal point and is parallel to the incident ray.* Every system of lenses having a common axis, whether the lenses are in contact with each other or at any distances apart, has two nodal points possessing the above property. An obvious deduction from this property is, that the image subtends the same angle at the second nodal point as the object subtends at the first. The second nodal point of the normal human eye is in the crystalline lens near the back, and the image of a distant object formed on the retina subtends the same angle at this point which the object subtends at the eye. The "line of collimation" of a telescope, which we have defined on page 1058 as "the line joining the cross to the optical centre of the object-glass," would be still more accurately defined as the line joining the cross to the second nodal point of the object-glass.

CHAPTER LXXIII.

COLOUR.

1067. Colour as a Property of Opaque Bodies.—A body which reflects (by irregular reflection) all the rays of the spectrum in equal proportion, will appear of the same colour as the light which falls upon it; that is to say, in ordinary cases, white or gray. But the majority of bodies reflect some rays in larger proportion than others, and are therefore coloured, their colour being that which arises from the mixture of the rays which they reflect. A body reflecting no light would be perfectly black. Practically, white, gray, and black differ only in brightness. A piece of white paper in shadow appears gray, and in stronger shadow black.

1068. Colour of Transparent Bodies.—A transparent body, seen by transmitted light, is coloured, if it is more transparent to some rays than to others, its colour being that which results from mixing the transmitted rays. No new ingredient is added by transmission, but certain ingredients are more or less completely stopped out.

Some transparent substances appear of very different colours according to their thickness. A solution of chloride of chromium, for example, appears green when a thin layer of it is examined, while a greater thickness of it presents the appearance of reddish brown. In such cases, different kinds of rays successively disappear by selective absorption, and the transmitted light, being always the sum of the rays which remain unabsorbed, is accordingly of different composition according to the thickness.

When two pieces of coloured glass are placed one behind the other, the light which passes through both has undergone a double process of selective absorption, and therefore consists mainly of those rays which are abundantly transmitted by both glasses; or to speak broadly, the colour which we see in looking through the combination

is not the sum of the colours of the two glasses, but their common part. Accordingly, if we combine a piece of ordinary red glass, transmitting light which consists almost entirely of red rays, with a piece of ordinary green glass, which transmits hardly any red, the combination will be almost black. The light transmitted through two glasses of different colour and of the same depth of tint, is always less than would be transmitted by a double thickness of either; and the colour of the transmitted light is in most cases a colour which occupies in the spectrum an intermediate place between the two given colours. Thus, if the two glasses are yellow and blue, the transmitted light will, in most cases, be green, since most natural yellows and blues when analysed by a prism show a large quantity of green in their composition. Similar effects are obtained by mixing coloured liquids.

1069. Colours of Mixed Powders.—"In a coloured powder, each particle is to be regarded as a small transparent body which colours light by selective absorption. It is true that powdered pigments when taken in bulk are extremely opaque. Nevertheless, whenever we have the opportunity of seeing these substances in compact and homogeneous pieces before they have been reduced to powder, we find them transparent, at least when in thin slices. Cinnabar, chromate of lead, verdigris, and cobalt glass are examples in point.

"When light falls on a powder thus composed of transparent particles, a small part is reflected at the upper surface; the rest penetrates, and undergoes partial reflection at some of the surfaces of separation between the particles. A single plate of uncoloured glass reflects $\frac{1}{25}$ of normally incident light; two plates $\frac{1}{13}$, and a large number nearly the whole. In the powder of such glass, we must accordingly conclude that only about $\frac{1}{25}$ of normally incident light is reflected from the first surface, and that all the rest of the light which gives the powder its whiteness comes from deeper layers. It must be the same with the light reflected from blue glass; and in coloured powders generally only a very small part of the light which they reflect comes from the first surface; it nearly all comes from beneath. The light reflected from the first surface is white, except when the reflection is metallic. That which comes from below is coloured, and so much the more deeply the further it has penetrated. This is the reason why coarse powder of a given material is more deeply coloured than fine, for the quantity of light returned at each successive reflection depends only on the number of reflections and not on the

thickness of the particles. If these are large, the light must penetrate so much the deeper in order to undergo a given number of reflections, and will therefore be the more deeply coloured.

"The reflection at the surfaces of the particles is weakened if we interpose between them, in the place of air, a fluid whose index of refraction more nearly approaches their own. Thus powders and pigments are usually rendered darker by wetting them with water, and still more with the more highly refracting liquid, oil.

"If the colours of powders depended only on light reflected from their first surfaces, the light reflected from a mixed powder would be the sum of the lights reflected from the surfaces of both. But most of the light, in fact, comes from deeper layers, and having had to traverse particles of both powders, must consist of those rays which are able to traverse both. The resultant colour therefore, as in the case of superposed glass plates, depends not on addition but rather on subtraction. Hence it is that a mixture of two pigments is usually much more sombre than the pigments themselves, if these are very unlike in the average refrangibility of the light which they reflect. Vermilion and ultramarine, for example, give a black-gray (showing scarcely a trace of purple, which would be the colour obtained by a true mixture of lights), each of these pigments being in fact nearly opaque to the light of the other." [1]

1070. Mixtures of Colours.—By the colour resulting from the mixture of two lights, we mean the colour which is seen when they both fall on the same part of the retina. Propositions regarding mixtures of colours are merely subjective. The only objective differences of colour are differences of refrangibility, or if traced to their source, differences of wave-frequency. All the colours in a pure spectrum are objectively simple, each having its own definite period of vibration by which it is distinguished from all others. But whereas, in acoustics, the quality of a sound as it affects the ear varies with every change in its composition, in colour, on the other hand, very different compositions may produce precisely the same visual impression. Every colour that we see in nature can be exactly imitated by an infinite variety of different combinations of elementary rays.

To take, for example, the case of white. Ordinary white light consists of all the colours of the spectrum combined; but any one of the elementary colours, from the extreme red to a certain point in yellowish green, can be combined with another elementary colour

[1] Translated from Helmholtz's *Physiological Optics*, § 20.

on the other side of green in such proportion as to yield a perfect imitation of ordinary white. The prism would instantly reveal the differences, but to the naked eye all these whites are completely undistinguishable one from another.

1071. Methods of Mixing Colours.—The following are some of the best methods of mixing colours (that is coloured lights):—

1. By combining reflected and transmitted light; for example, by looking at one colour through a piece of glass, while another colour is seen by reflection from the near side of the glass. The lower sash of a window, when opened far enough to allow an arm to be put through, answers well for this purpose. The brighter of the two coloured objects employed should be held inside the window, and seen by reflection; the second object should then be held outside in such a position as to be seen in coincidence with the image of the first. As the quantity of reflected light increases with the angle of incidence, the two colours may be mixed in various proportions by shifting the position of the eye. This method is not however adapted to quantitative comparison, and can scarcely be employed for combining more than two colours.

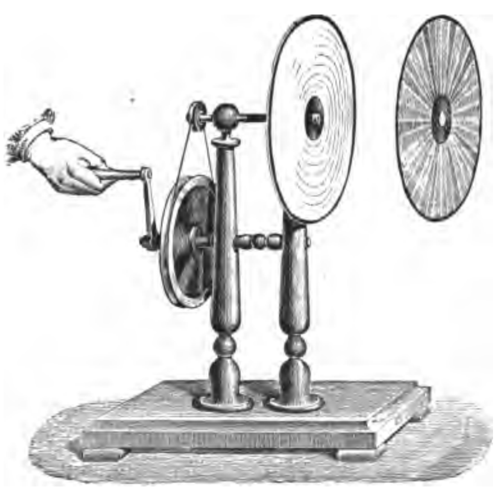


Fig. 768.—Rotating Disc.

2. By employing a rotating disc (Fig. 768) composed of differently coloured sectors. If the disc be made to revolve rapidly, the sectors will not be separately visible, but their colours will appear blended into one on account of the persistence of visual impressions. The proportions can be varied by varying the sizes of the sectors. Coloured discs of paper, each having a radial slit, are very convenient for this purpose, as any moderate number of such discs can be combined, and the sizes of the sectors exhibited can be varied at pleasure.

The mixed colour obtained by a rotating disc is to be regarded as

a *mean* of the colours of the several sectors—a mean in which each of these colours is assigned a weight proportional to the size of its sector. Thus, if the 360 degrees which compose the entire disc consist of 100° of red paper, 100° of green, and 160° of blue, the intensity of the light received from the red when the disc is rotating will be only $\frac{10}{36}$ of that which would be received from the red sector when seen at rest; and the total effect on the retina is represented by $\frac{10}{36}$ of the intensity of the red, *plus* $\frac{10}{36}$ of the intensity of the green, *plus* $\frac{16}{36}$ of the intensity of the blue; or if we denote the colours of the sectors by their initial letters, the effect may be symbolized by the formula $\frac{10\,R+10\,G+16\,B}{36}$. Denoting the resultant colour by C, we have the symbolic equation

$$10\,R+10\,G+16\,B=36\,C;$$

and the resultant colour may be called the mean of 10 parts of red, 10 of green, and 16 of blue. Colour-equations, such as the above, are frequently employed, and may be combined by the same rules as ordinary equations.

3. By causing two or more spectra to overlap. We thus obtain mixtures which are the *sums* of the overlapping colours.

If, in the experiment of § 1048, we employ, instead of a single straight slit, a pair of slits meeting at an angle, so as to form either an X or a V, we shall obtain mixtures of all the simple colours two and two, since the coloured images of one of the slits will cross those of the other. The display of colours thus obtained upon a screen is exquisitely beautiful, and if the eye is placed at any point of the image (for example, by looking through a hole in the screen), the prism will be seen filled with the colour which falls on this point.

1072. Experiments of Helmholtz and Maxwell.—Helmholtz, in an excellent series of observations of mixtures of simple colours, employed a spectroscope with a V-shaped slit, the two strokes of the V being at right angles to one another; and by rotating the V he was able to diminish the breadth and increase the intensity of one of the two spectra, while producing an inverse change in the other. To isolate any part of the compound image formed by the two overlapping spectra, he drew his eye back from the eye-piece, so as to limit his view to a small portion of the field.

But the most effective apparatus for observing mixtures of simple colours is one devised by Professor Clerk Maxwell, by means of which any two or three colours of the spectrum can be combined in

any required proportions. In principle, this method is nearly equivalent to looking through the hole in the screen in the experiment above described.

Let P (Fig. 769) be a prism, in the position of minimum deviation; L a lens; E and R conjugate foci for rays of a particular refrangibility, say red; E and V conjugate foci for rays of another given refrangibility, say violet. If a slit is opened at R, an eye at E will receive only red rays, and will see the lens filled with red light. If this slit be closed, and a slit opened at V, the eye, still placed at E, will see the lens filled with violet light. If both slits be opened, it will see the lens filled with a uniform mixture of the two lights; and if a third slit be opened, between R and V, the lens will be seen filled with a mixture of three lights.

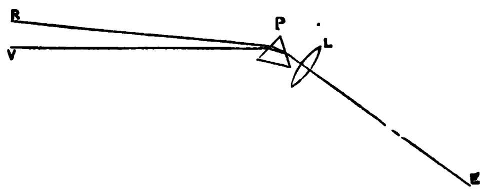


Fig. 769.—Principle of Maxwell's Colour-box.

Again, from the properties of conjugate foci, if a slit is opened at E, its spectral image will be formed at R V, the red part of it being at R, and the violet part at V.

The apparatus was inclosed in a box painted black within. There was a slit fixed in position at E, and a frame with three movable slits at R V. When it was desired to combine colours from three given parts of the spectrum, specified by reference to Fraunhofer's lines, the slit E was first turned towards the light, giving a real spectrum in the plane R V, in which Fraunhofer's lines were visible, and the three movable slits were set at the three specified parts of the spectrum. The box was then turned end for end, so that light was admitted (reflected from a large white screen placed in sunshine) at the movable slits, and the observer, looking in at the slit E, saw the resultant colour.

1073. Results of Experiment.—The following are some of the principal results of experiments on the mixture of coloured lights:—

1. Lights which appear precisely alike to the naked eye yield identical results in mixtures; or employing the term *similar* to express apparent identity as judged by the naked eye, *the sums of similar lights are themselves similar.* It is by reason of this physical fact, that colour-equations yield true results when combined according to the ordinary rules of elimination.

In the strict application of this rule, the same observer must be the judge of similarity in the different cases considered. For

2. Colours may be similar as seen by one observer, and dissimilar as seen by another; and in like manner, colours may be similar as seen through one coloured glass, and dissimilar as seen through another. The reason, in both cases, is that selective absorption depends upon real composition, which may be very different for two merely similar lights. Most eyes are found to exhibit selective absorption of a certain kind of elementary blue, which is accordingly weakened before reaching the retina.

3. Between any four colours, given in intensity as well as in kind, one colour-equation subsists; expressing the fact that, when we have the power of varying their intensities at pleasure, there is one definite way of making them yield a *match*, that is to say, a pair of similar colours. Any colour can therefore be completely specified by three numbers, expressing its relation to three arbitrarily selected colours. This is analogous to the theorem in statics that a force acting at a given point can be specified by three numbers denoting its components in three arbitrarily selected directions.

4. Between any five colours, given in intensity as well as in kind, a match can be made in one definite way by taking means;[1] for example, by mounting the colours on two rotating discs. If we had the power of illuminating one disc more strongly than the other in any required ratio, four colours would be theoretically sufficient; and we can, in fact, do what is nearly equivalent to this, by employing black as one of our five colours. Taking means of colours is analogous to finding centres of gravity. In following out the analogy, a colour given in kind merely must be represented by a material point given in position merely, and the intensity of the colour must be represented by the mass of the material point. The means of two given colours will be represented by points in the line joining two given points. The means of three given colours will be represented by points lying within the triangle formed by joining three given points, and the means of four given colours will be represented by points within a tetrahedron whose four corners are given. When we have five colours given, we have five points given, and of these generally no four will lie in one plane. Call them A, B, C, D, E.

[1] Propositions 4 and 5 are not really independent, but represent different aspects of one physical (or rather physiological) law.

Then if E lies within the tetrahedron ABCD, we can make the centre of gravity of A, B, C, and D coincide with E, and the colour E can be matched by a mean of the other four colours.

If E lies outside the tetrahedron, it must be situated at a point from which either one, two, or three faces are visible (the tetrahedron being regarded as opaque).

If only one face is visible, let it be BCD; then the point where the straight line EA cuts BCD is the match; for it is a mean of E and A, and is also a mean of B, C, and D.

If two faces are visible, let them be ACD and BCD; then the intersection of the edge CD with the plane EAB is the match.

If three faces are visible, let them be the three which meet at A; then A is the match, for it lies within the tetrahedron EBCD.

With six given colours, combined five at a time, six different matches can be made, and six colour-equations will thus be obtained, the consistency of which among themselves will be a test of the accuracy both of theory and observation, as only three of the six can be really independent. Experiments which have been conducted on this plan have given very consistent results.

1074. Cone of Colour.—All the results of mixing colours can be represented geometrically by means of a cone or pyramid within which all possible colours will have their definite places. The vertex will represent total blackness, or the complete absence of light; and colours situated on the same line passing through the vertex will differ only in intensity of light. Any cross-section of the cone will contain all colours, except so far as intensity is concerned, and the colours residing on its perimeter will be the colours of the spectrum ranged in order, with purple to fill up the interval between violet and red. It appears from Maxwell's experiments, that the true form of the cross-section is approximately triangular;[1] with red, green, and violet at the three corners. When all the colours have been assigned their proper places in the cone, a straight line joining any two of them passes through colours which are means of these two; and if two lines are drawn from the vertex to any two colours, the parallelogram constructed on these two lines will have at its further corner the colour which is the sum of these two colours. A certain axial line of the cone will contain

[1] The shape of the triangle is a mere matter of convenience, not involving any question of fact.

white or gray at all points of its length, and may be called the *line of white.*

It is convenient to distinguish three qualities of colour which may be called *hue, depth,* and *brightness. Brightness* or *intensity* of light is represented by distance from the vertex of the cone. *Depth* depends upon angular distance from the line of white, and is the same for all points on the same line through the vertex. *Paleness* or *lightness* is the opposite of depth, and is measured by angular nearness to the line of white. *Hue* or *tint* is that which is often *par excellence* termed colour. If we suppose a plane, containing the line of white, to revolve about this line as axis, it will pass successively through different tints; and in any one position it contains only two tints, which are separated from each other by the line of white, and are complementary.

Red is complementary to			Bluish green.
Orange	„	„	Sky blue.
Yellow	„	„	Violet blue.
Greenish yellow		„	Violet.
Green	„	„	Pink.

Any two colours of complementary tint give white, when mixed in proper proportions; and any three colours can be mixed in such proportions as to yield white, if the triangle formed by joining them is pierced by the line of white.

Every colour in nature, except purple, is similar to a colour of the spectrum either pure or diluted with gray; and all purples are similar to mixtures of red and blue with or without dilution. Brown can be imitated by diluting orange with dark gray. The orange and yellow of the spectrum can themselves be imitated by adding together red and green.

1075. Three Primary Colour-sensations.—All authorities are now agreed in accepting the doctrine, first propounded by Dr. Thomas Young, that there are three elements of colour-sensation; or, in other words, three distinct physiological actions, which, by their various combinations, produce our various sensations of colour. Each is excitable by light of various wave-lengths lying within a wide range, but has a maximum of excitability for a particular wave-length, and is affected only to a slight degree by light of wave-length very different from this. The cone of colour is theoretically a triangular pyramid, having for its three edges the colours which correspond to these three wave-lengths; but it is probable that we cannot obtain

one of the three elementary colour-sensations quite free from admixture of the other two, and the edges of the pyramid are thus practically rounded off. One of these sensations is excited in its greatest purity by the green near Fraunhofer's line *b*, another by the extreme red, and the third by the extreme violet.

Helmholtz ascribes these three actions to three distinct sets of nerves, having their terminations in different parts of the thickness of the retina—a supposition which aids in accounting for the approximate achromatism of the eye, for the three sets of nerve-terminations may thus be at the proper distances for receiving distinct images of red, green, and violet respectively, the focal length of a lens being shorter for violet than for red.

Light of great intensity, whatever its composition, seems to produce a considerable excitement of all three elements of colour-sensation. If a spectroscope, for example, be directed first to the clouds and then to the sun, all parts of the spectrum appear much paler in the latter case than in the former.

The popular idea that red, yellow, and blue are the three primaries, is quite wrong as regards mixtures of lights or combinations of colour-sensations. The idea has arisen from facts observed in connection with the mixture of pigments and the transmission of light through coloured glasses. We have already pointed out the true interpretation of observations of this nature, and have only now to add that in attempting to construct a theory of the colours obtained by mixtures of pigments, the law of substitution of *similars* cannot be employed. Two pigments of *similar* colour will not in general give the same result in mixtures.

1076. Accidental Images.—If we look steadily at a bright stained-glass window, and then turn our eyes to a white wall, we see an image of the window with the colours changed into their complementaries. The explanation is that the nerves which have been strongly exercised in the perception of the bright colours have had their sensibility diminished, so that the balance of action which is necessary to the sensation of white no longer exists, but those elements of sensation which have not been weakened preponderate. The subjective appearances arising from this cause are called *negative accidental images.* Many well-known effects of contrast are similarly explained. White paper, when seen upon a background of any one colour, often appears tinged with the complementary colour; and stray beams of sunlight entering a room shaded with

yellow holland blinds, produce blue streaks when they fall upon a white tablecloth.

In some cases, especially when the object looked at is painfully bright, there is a *positive* accidental image; that is, one of the same colour as the object; and this is frequently followed by a negative image. A positive accidental image may be regarded as an extreme instance of the persistence of impressions.

1077. Colour-blindness.—What is called colour-blindness has been found, in every case which has been carefully investigated, to consist in the absence of the elementary sensation corresponding to red. To persons thus affected the solar spectrum appears to consist of two decidedly distinct colours, with white or gray at their place of junction, which is a little way on the less refrangible side of the line F. One of these two colours is doubtless nearly identical with the normal sensation of blue or violet. It attains its maximum about midway between F and G, and extends beyond G as far as the normally visible spectrum. The other colour extends a considerable distance into what to normal eyes is the red portion of the spectrum, attaining its maximum about midway between D and E, and becoming deeper and more faint till it vanishes at about the place where to normal eyes crimson begins. The scarlet of the spectrum is thus visible to the colour-blind, not as scarlet but as a deep dark colour, perhaps a kind of dark green, orange and yellow as brighter shades of the same colour, while bluish-green appears nearly white.

It is obvious from this account that what is called "colour-blindness" should rather be called *dichroic vision,* normal vision being distinctively designated as *trichroic.* To the dichroic eye any colour can be matched by a mixture of yellow and blue, and a match can be made between any three (instead of four) given colours. Objects which have the same colour to the trichroic eye have also the same colour to the dichroic eye.

1078. Colour and Musical Pitch.—As it is completely established that the difference between the colours of the spectrum is a difference of vibration-frequency, there is an obvious analogy between colour and musical pitch; but in almost all details the relations between colours are strikingly different from the relations between sounds.

The compass of visible colour, including the lavender rays which lie beyond the violet, and are perhaps visible not in themselves,

but by the fluorescence which they produce on the retina, is, according to Helmholtz, about an octave and a fourth; but if we exclude the lavender, it is almost exactly an octave. Attempts have been made to compare the successive colours of the spectrum with the notes of the gamut; but much forcing is necessary to bring out any trace of identity, and the gradual transitions which characterize the spectrum, and constitute a feature of its beauty, are in marked contrast to the transitions *per saltum* which are required in music.

CHAPTER LXXIV.

WAVE THEORY OF LIGHT.

1079. Principle of Huygens.[1]—The propagation of waves, whether of sound or light, is a propagation of energy. Each small portion of the medium experiences successive changes of state, involving changes in the forces which it exerts upon neighbouring portions. These changes of force produce changes of state in these neighbouring portions, or in such of them as lie on the forward side of the wave, and thus a disturbance existing at any one part is propagated onwards.

Let us denote by the name *wave-front* a continuous surface drawn through particles which have the same phase; then each wave-front advances with the velocity of light, and each of its points may be regarded as a secondary centre from which disturbances are continually propagated. This mode of regarding the propagation of light is due to Huygens, who derived from it the following principle, which lies at the root of all practical applications of the undulatory theory: *The disturbance at any point of a wave-front is the resultant* (given by the parallelogram of motions) *of the separate disturbances which the different portions of the same wave-front in any one of its earlier positions, would have occasioned if acting singly.* This principle involves the physical fact that rays of light are not affected by crossing one another; and its truth, which has been experimentally tested by a variety of consequences, must be taken as an indication that the amplitudes of luminiferous vibrations are infinitesimal in comparison with the wave-lengths. A similar law applies to the resultant of small disturbances generally, and is called by writers on dynamics the law of "superposition of small motions." It is analogous to the arithmetical principle that, when a and b are very small fractions, the product of $1+a$ and $1+b$ may be identified with

[1] For the spelling of this name see remarks by Lalande, *Mémoires de l'Académie*, 1773.

$1+a+b$; the term $a\,b$, which represents the mutual influence of two small changes, being negligible in comparison with the sum $a+b$ of the small changes themselves.

1080. Explanation of Rectilinear Propagation.—In a medium in which light travels with the same velocity in all parts and in all directions, the waves propagated from any point will be concentric spheres, having this point for centre; and the lines of propagation, in other words the rays of light, will be the radii of these spheres. It can in fact be shown that the only part of one of these waves which needs to be considered, in computing the resultant disturbance of an external point, is the part which lies directly between this external point and the centre of the sphere. The remainder of the wave-front can be divided into small parts, each of which, by the mutual interference of its own subdivisions, gives a resultant effect of zero at the given point. We express these properties by saying that *in a homogeneous and isotropic medium the wave-surface is a sphere, and the rays are normal to the wave-fronts.* This class of media includes gases, liquids, crystals of the cubic system, and well-annealed glass.

If a medium be homogeneous but not isotropic, disturbances emanating from a point in it will be propagated in waves which will retain their form unchanged as they expand in receding from their source, but this form will not generally be spherical. The rays of light in such a medium will be straight, proceeding directly from the centre of disturbance, and any one ray will cut all the wave-fronts at the same angle; but this angle will generally be different for different rays. In this case, as in the last, the disturbance produced at any point may be computed by merely taking into account that small portion of a wave-front which lies directly between the given point and the source,—in other words, which lies on or very near to the ray which traverses the given point.

A disturbance in such a medium usually gives rise to two sets of waves, having two distinct forms, and these remarks apply to each set separately.

The tendency of the different parts of a wave-front to propagate disturbances in other directions besides the single one to which such propagation is usually confined, is manifested in certain phenomena which are included under the general name of *diffraction.*

The only wave-fronts with which it is necessary to concern ourselves are those which belong to waves emanating from a single

point,—that is to say, either from a surface really very small, or from a surface which, by reason of its distance, subtends a very small solid angle at the parts of space considered.

1081. Application to Refraction.—When waves are propagated from one medium into another, the principle of Huygens leads to the following construction:—

Let A E (Fig. 770) represent a portion of the surface of separation between two media, and A B a portion of a wave-front in the first medium; both portions being small enough to be regarded as plane.

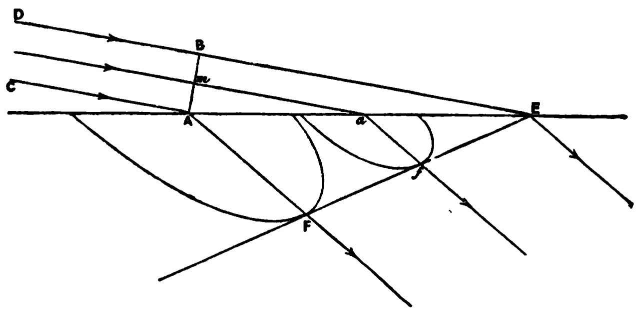


Fig. 770.—Huygens' Construction for Wave-front.

Then straight lines C A, D B E, normal to the wave-front, represent rays incident at A and E. From A as centre, describe a wave-surface, of such dimensions that light emanating from A would reach this surface in the same time in which light in air travels the distance B E, and draw a tangent plane (perpendicular to the plane of incidence) through E to this surface. Let F be the point of contact (which is not necessarily in the plane of incidence). Then the tangent plane E F is a wave-front in the second medium, and A F is a ray in the second medium; for it can be shown that disturbances propagated from all points in the wave-front A B will just have reached E F when the disturbance propagated from B has reached E. For example, a ray proceeding from m, the middle point of the line A B, will exhaust half the time in travelling to the middle point a of A E, and the remaining half in travelling through af, equal and parallel to half of A F.

When the wave-surfaces in both media are spherical, the planes of incidence and refraction A B E, A F E coincide, the angle B A E (Fig. 771) between the first wave-front and the surface of separation is the same as the angle between the normals to these surfaces, that

is to say, is the angle of incidence; and the angle A E F between the surface of separation and the second wave-front is the angle of refraction. The sine of the former is $\frac{BE}{EA}$, and the sine of the latter is $\frac{AF}{EA}$. The ratio $\frac{\sin i}{\sin r}$ is therefore $\frac{BE}{AF}$. But B E and A F are the distances travelled in the same time in the two media. Hence the sines of the angles of incidence and refraction are directly as the velocities of propagation of the incident and refracted light. The *relative index* of refraction from one medium into another is therefore the *ratio of the velocity of light in the first medium to its velocity in the second;* and *the absolute index of refraction of any medium is inversely as the velocity of light in that medium.*

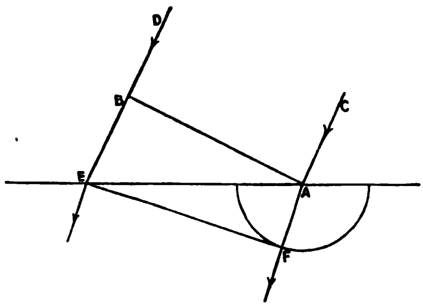


Fig. 771.—Wave-front in Ordinary Refraction.

1082. Application to Reflection.—The explanation of reflection is precisely similar. Let C A, D E (Fig. 772) be parallel rays incident at A and E; A B the wave-front. As the successive points of the wave-front arrive at the reflecting surface, hemispherical waves diverge from the points of incidence; and by the time that B reaches E, the wave from A will have diverged in all directions to a distance equal to B E. If then we describe in the plane of incidence a semicircle, with centre A and radius equal to B E, the tangent E F to this semicircle will be the wave-front of the reflected light, and A F will be the reflected ray corresponding to the incident ray C A. From the equality of the right-angled triangles A B E, E F A, it is evident that the angles of incidence and reflection are equal.

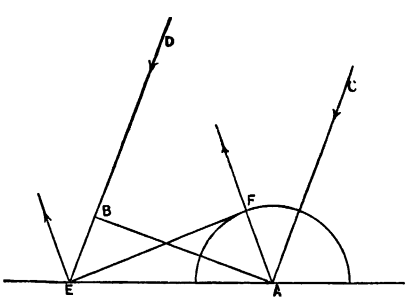


Fig. 772.—Wave-front in Reflection.

1083. Newtonian Explanation of Refraction.—In the Newtonian theory, the change of direction which a ray experiences at the bound-

ing surface of two media, is attributed to the preponderance of the attraction of the denser medium upon the particles of light. As the resultant force of this attraction is normal to the surface, the tangential component of velocity remains unchanged, and the normal component is increased or diminished according as the incidence is from rare to dense or from dense to rare. Let μ denote the relative index of refraction from rare to dense. Let v, v' be the velocities of light in the rarer and denser medium respectively, and i, i' the angles which the rays in the two media make with the normal. Then the tangential components of velocity in the two media are $v \sin i, v' \sin i'$ respectively, and these by the Newtonian theory are equal; whence $\frac{v'}{v}=\frac{\sin i}{\sin i'}=\mu$; whereas according to the undulatory theory $\frac{v'}{v}=\frac{1}{\mu}$. In the Newtonian theory, the velocity of light in any medium is directly as the absolute index of refraction of the medium; whereas, in the undulatory theory, the reverse rule holds.

The main design of Foucault's experiment with the rotating mirror (§ 942), in its original form, was to put these opposite conclusions to the test of direct experiment. For this purpose it was not necessary to determine the velocity of the rotating mirror, since it affected both the observed displacements alike. The two images were seen in the same field of view, and were easily distinguished by the greenness of the water-image. In every trial the water-image was more displaced than the air-image, indicating longer time and slower velocity; and the measurements taken were in complete accordance with the undulatory theory, while the Newtonian theory was conclusively disproved.

1084. Principle of Least Time.—The path by which light travels from one point to another is in the generality of cases that which occupies least time. For example, in ordinary cases of reflection (except from very concave[1] surfaces), if we select any two points, one on the incident and the other on the reflected ray, the sum of their distances from the point of incidence is less than the sum of their distances from any neighbouring point on the reflecting surface. In this case, since only one medium is concerned, distance is proportional to time. When a ray in air is refracted into water, if we select any two points,

[1] Suppose an ellipse described, having the two selected points for foci, and passing through the point of incidence. If the curvature of the reflecting surface in the plane of incidence is greater than the curvature of this ellipse, the length of the path is a maximum, if less, a minimum. This follows at once from the constancy of the sum of the focal distances in an ellipse.

one on the incident and the other on the refracted ray, and call their distances from any point of the refracting surface s, s' respectively, and the velocities of propagation in the two media v, v', then the sum of $\frac{s}{v}$ and $\frac{s'}{v'}$ is generally less when s and s' are measured to the point of incidence than when they are measured to any neighbouring point on the surface. $\frac{s}{v}$ is evidently the time of going from the first point to the refracting surface, and $\frac{s'}{v'}$ the time from the refracting surface to the second point.

The proposition as above enunciated admits of certain exceptions, the time being sometimes a maximum instead of a minimum. The really essential condition (which is fulfilled in both these opposite cases) is that all points on a small area surrounding the point of incidence give sensibly *the same time.* The component waves sent from all parts of this small area will be in the same phase, and will propagate a ray of light by their combined action.

When the two points considered are conjugate foci, and there is no aberration, this condition must be fulfilled by all the rays which pass through both; and the *time of travelling from one focus to the other is the same for all the rays.* Spherical waves diverging from one focus will, after incidence, become spherical waves converging to or diverging from its conjugate focus. An effect of this kind can be beautifully exhibited to the eye by means of an elliptic dish containing mercury. If agitation is produced at one focus of the ellipse by dipping a small rod into the liquid at this point, circular waves will be seen to converge towards the other focus. A circular dish exhibits a similar result somewhat imperfectly; waves diverging from a point near the centre will be seen to converge to a point symmetrically situated on the other side of the centre.

When the second point lies on a caustic surface formed by the reflection or refraction of rays emanating from the first point, all points on an area of sensible magnitude in the neighbourhood of the point of incidence would give sensibly the same time of travelling as the actual point of incidence, so that the light which traverses a point on a caustic may be regarded as coming from an area of sensible magnitude instead of (as in the case of points not on the caustic) an excessively small area. An eye placed at a point on a caustic will see this portion of the surface filled with light.

As the velocity of light is inversely proportional to the index of

refraction μ, the time of travelling a distance s with constant velocity may be represented by μs, and if a ray of light passes from one point to another by a crooked path, made up of straight lines $s_1, s_2, s_3, \ldots$ lying in media whose absolute indices are $\mu_1, \mu_2, \mu_3, \ldots$, the expression $\mu_1 s_1 + \mu_2 s_2 + \mu_3 s_3 + \ldots$ represents the time of passage. This expression, which may be called *the sum of such terms as* μs, must therefore fulfil the above condition; that is to say, the points of incidence on the surfaces of separation must be so situated that this sum either remains absolutely constant when small changes are supposed to be made in the positions of these points, or else retains that approximate constancy which is characteristic of maxima and minima. Conversely, all lines from a luminous point which fulfil this condition, will be paths of actual rays.

1085. Terrestrial Refraction.[1]—The atmosphere may be regarded as homogeneous when we confine our attention to small portions of it, and hence it is sensibly true, in ordinary experiments where no great distances are concerned, that rays of light in air are straight, just as it is true in the same limited sense that the surface of a liquid at rest is a horizontal plane. The surface of an ocean is not plane, but approximately spherical, its curvature being quite sensible in ordinary nautical observations, where the distance concerned is merely that of the visible sea-horizon; and a correction for curvature is in like manner required in observing levels on land. If the observer is standing on a perfectly level plain, and observing a distant object at precisely the same height as his eye above the plain, it will appear to be below his eye, for a horizontal *plane* through his eye will pass above it, since a perfectly level *plain* is not *plane*, but shares in the general curvature of the earth. It is easily proved that the apparent depression due to this cause is half the angle between the verticals at the positions of the observer and of the object observed. But experience has shown that this apparent depression is to a considerable extent modified by an opposite disturbing cause, called *terrestrial refraction.* When the atmosphere is in its normal condition, a ray of light from the object to the observer is not straight, but is slightly concave downwards.

This curvature of a nearly horizontal ray is not due to the curvature of the earth and of the layers of equal density in the earth's atmosphere, as is often erroneously supposed, but would still exist,

[1] For the leading idea which is developed in §§ 1085–1087, the Editor is indebted to suggestions from Professor James Thomson.

and with no sensible change in its amount, if the earth's surface were plane, and the directions of gravity everywhere parallel. It is due to the fact that light travels faster in the rarer air above than in the denser air below, so that time is saved by deviating slightly to the upper side of a straight course. The actual amount of curvature (as determined by surveying) is from $\frac{1}{2}$ to $\frac{1}{10}$ of the curvature of the earth; that is to say, the radius of curvature of the ray is from 2 to 10 times the earth's radius.

1086. Calculation of Curvature of Ray.—In order to calculate the radius of curvature from physical data, it is better to approach the subject from a somewhat different point of view.

The wave-fronts of a ray in air are perpendicular to the ray; and if the ray is nearly horizontal, its wave-fronts will be nearly vertical. If two of these wave-fronts are produced downwards until they meet, the distance of their intersection from the ray will be the radius of curvature. Let us consider two points on the same wave-front, one of them a foot above the other; then the upper one being in rarer air will be advancing faster than the lower one, and it is easily shown that the difference of their velocities is to the velocity of either, as 1 foot is to the radius of curvature.

Put ρ for the radius of curvature in feet, v and $v+\delta v$ for the two velocities, μ and $\mu-\delta\mu$ for the indices of refraction of the air at the two points. Then we have

$$\frac{1}{\rho} = \frac{\delta v}{v} = \frac{\delta \mu}{\mu} = \delta\mu \text{ nearly.} \qquad (1)$$

Now it has been ascertained, by direct experiment, that the value of $\mu-1$ for air, within ordinary limits of density, is sensibly proportional to the density (even when the temperature varies), and is ·0002943 or $\frac{1}{3400}$ at the density corresponding to the pressure 760^{mm}, (at Paris) and temperature 0° C. The difference of density at the two points considered, supposing them both to be at the same temperature, will be to the density of either as 1 foot is to the "height of the homogeneous atmosphere" in feet, which call H (§ 211). Then $\frac{\delta\mu}{\mu-1}$ will be $\frac{1}{\mathrm{H}}$, and the value of $\frac{1}{\rho}$ in (1) may be written

$$\frac{1}{\rho} = \frac{\delta\mu}{\mu-1}(\mu-1) = \frac{1}{\mathrm{H}}(\mu-1) = \frac{1}{\mathrm{H}}\,\frac{1}{3400}. \qquad (2)$$

Hence ρ is 3400 times the height of the homogeneous atmosphere. But this height is about 5 miles, or $\frac{1}{800}$ of the earth's radius. The

value of ρ is therefore about $4\frac{1}{4}$ radii of the earth. This is on the assumptions that the barometer is at 760^{mm}, the thermometer at 0° C., and that there is no change of temperature in ascending. If we depart from these assumptions, we have the following consequences:—

I. If the barometer is at any other height, the factor $\frac{1}{H}$ remains unaltered, and the other factor $\mu - 1$ varies directly as the pressure.

II. If the temperature is t° Centigrade, H is changed in the direct ratio of $1 + \alpha t$, α denoting the coefficient of expansion. The first factor $\frac{1}{H}$ is therefore changed in the inverse ratio of $1 + \alpha t$. The second factor is changed in the same ratio. The curvature of the ray therefore varies inversely as $(1 + \alpha t)^2$.

III. Suppose the temperature decreases upwards at the rate of $\frac{1}{n}$ of a degree Centigrade per foot. The expansion due to $\frac{1}{n}$ of a degree Centigrade is $\frac{1}{273n}$. The first factor $\frac{\delta\mu}{\mu - 1}$, or $\frac{\text{difference of density}}{\text{density}}$, will therefore become $\frac{1}{H} - \frac{1}{273n}$, which, if we put $n = 540$ (corresponding to 1° Fahr. in 300 feet), and reckon H as 26,000, is approximately $\frac{1}{26000} - \frac{1}{147000}$ or $\frac{1}{H}\left(1 - \frac{1}{6}\right)$. The second factor of the expression for $\frac{1}{\rho}$ is unaffected. It appears, then, that decrease of temperature upwards at the rate of 1° C. in 540 feet, or 1° F. in 300 feet (which is the generally-received average), makes the curvature of the ray five-sixths of what it would be if the temperature were uniform.[1]

Combining this correction with correction II., it appears that, with a mean temperature of 10° C. or 50° F., and barometer at 760^{mm}, the curvature of a nearly horizontal ray (taking the earth's curvature as unity) is

$$\frac{1}{4\frac{1}{4}} \times \left(\frac{273}{283}\right)^2 \times \frac{5}{6} = \frac{1}{5 \cdot 5} \text{ nearly.}$$

This is in perfect agreement with observation, the received average (obtained as an empirical deduction from observation) being $\frac{1}{5}$ or $\frac{1}{6}$.

1087. Curvature of Inclined Rays.—Thus far we have been treating of nearly horizontal rays. To adapt our formula for $\frac{1}{\rho}$ ((2) § 1086) to the case of an oblique ray, we have merely to multiply it by $\cos\theta$,

[1] If the temperature decreases upwards at the rate of 1° C. in n feet, or 1° F. in n' feet, the first factor of the expression for $\frac{1}{\rho}$ (which would be $\frac{1}{H}$ at uniform temperature) becomes approximately $\frac{1}{H}\left(1 - \frac{96}{n}\right)$ or $\frac{1}{H}\left(1 - \frac{53}{n'}\right)$.

θ denoting the inclination of the ray to the horizontal, or the inclination of the wave-front to the vertical. For, if we still compare two points a foot apart, on the same wave-front, and in the same vertical plane with each other and with the ray, their difference of height will be the product of 1 foot by cos θ, and $\frac{\delta v}{v}$ will therefore be less than before in the ratio cos θ.

Hence it can be shown that the earth's curvature, so far from being the cause of terrestrial refraction, rather tends in ordinary circumstances to diminish it, by increasing the average obliquity of a ray joining two points at the same level.

The general formula for the curvature of a ray (lying in a vertical plane) at any point in its length, may be written

$$\begin{aligned}\frac{1}{\rho} &= \frac{1}{H}\left(1-\frac{96}{n}\right)(\mu-1)\cos\theta \\ &= \frac{1}{H}\left(1-\frac{53}{n'}\right)(\mu-1)\cos\theta,\end{aligned} \qquad (3)$$

n denoting the number of feet of ascent which give a decrease of 1° C., and n' the number of feet which give a decrease of 1° F. The unit of length for H and ρ may be anything we please.

1088. Astronomical Refraction.—Astronomical refraction, in virtue of which stars appear nearer the zenith than they really are, can be reduced to these principles; but it is simpler, in the case of stars not more than 70° or 80° from the zenith, to regard the earth and the layers of equal density in the atmosphere as plane, and to assume (§ 993) that the final result is the same as if the rays from the star were refracted at once out of vacuum into the horizontal stratum of air in which the observer's eye is situated. If z be the apparent and $z+h$ the true zenith distance, we shall thus have

$$\begin{aligned}\mu \sin z &= \sin(z+h) \\ &= \sin z \cos h + \cos z \sin h \\ &= \sin z + h \cos z, \text{ nearly,}\end{aligned}$$

whence

$$h = (\mu - 1)\tan z.$$

1089. Mirage.—An appearance, as of water, is frequently seen in sandy deserts, where the soil is highly heated by the sun. The observer sees in the distance the reflection of the sky and of terrestrial objects, as in the surface of a calm lake. This phenomenon, which is called *mirage*, is explained by the heating and consequent rarefaction of the air in contact with the hot soil. The density,

within a certain distance of the ground, increases upwards, and rays traversing this portion of the air are bent upwards (Fig. 773), in accordance with the general rule that the concavity must be turned towards the denser side. Rays which were descending at a very slight inclination before entering this stratum of air may have their direction so much changed as to be bent up to an observer's eye, and the change of direction will be greatest for those rays which have

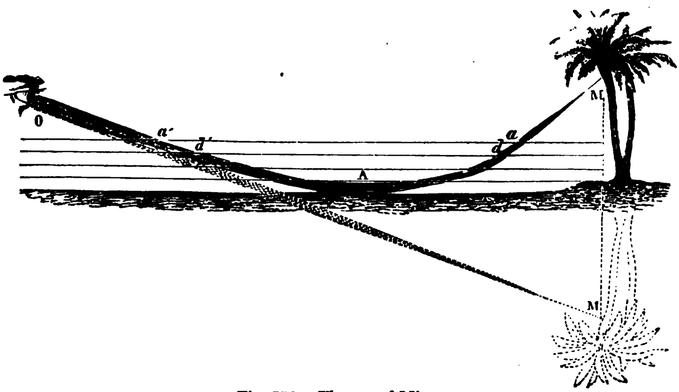


Fig. 773.—Theory of Mirage.

descended lowest; for these will not only have travelled for the greatest distance in the stratum, but will also have travelled through that part of it in which the change of density is most rapid. Hence, if we trace a pencil of rays from the observer's eye, we shall find that those of them which lie in the same vertical plane cross each other in traversing this stratum, and thus produce inverted images. If the stratum is thin in comparison with the height of the observer's eye, the appearance presented will be nearly equivalent to that produced by a mirror, while the objects thus reflected are also seen erect by higher rays which have not descended into the stratum where this action occurs.

A kind of inverted mirage is often seen across masses of calm water, and is called *looming;* images of distant objects, such as ships or hills, being seen in an inverted position immediately over the objects themselves. The explanation just given of the mirage of the desert will apply to this phenomenon also, if we suppose at a certain height, greater than that of the observer's eye, a layer of rapid transition from colder and denser air below to warmer and rarer air above.

An appearance similar to mirage may be obtained by gently depositing alcohol or methylated spirit upon water in a vessel with plate-glass sides. The spirit, though lighter, has a higher index of refraction than the water, and rays traversing the layer of transition are bent upwards. This layer accordingly behaves like a mirror when looked at very obliquely by an eye above it.[1]

1090. Curved Rays of Sound.—The reasoning of §§ 1084, 1086 can be applied, with a slight modification, to the propagation of sound.

Sound travels faster in warm than in cold air. On calm sunny afternoons, when the ground has become highly heated by the sun's rays, the temperature of the air is much higher near the ground than at moderate heights; hence sound bends upwards, and may thus become inaudible to observers at a distance by passing over their heads. On the other hand, on clear calm nights the ground is cooled by radiation to the sky, and the layers of air near the ground are colder than those above them; hence sound bends downwards, and may thus, by arching over intervening obstacles, become audible at distant points, which it could not reach by rectilinear propagation. This influence of temperature, which was first pointed out by Professor Osborne Reynolds, is one reason why sound from distant sources is better heard by night than by day.

A similar effect of wind had been previously pointed out by Professor Stokes. It is well known that sound is better heard with the wind than against it. This difference is due to the circumstance that wind is checked by friction against the earth, and therefore increases in velocity upwards. Sound travelling with the wind, therefore, travels fastest above, and sound travelling against the wind travels fastest below, its actual velocity being in the former case the sum, and in the latter the difference, of its velocity in still air and the velocity of the wind. The velocity of the wind is so much less than that of sound, that if uniform at all heights its influence on audibility would scarcely be appreciable.

1091. Calculation.—To calculate the curvature of a ray of sound due to variation of temperature with height, we may employ, as in § 1086, the formula $\frac{1}{\rho} = \frac{\delta v}{v}$, where δv denotes the difference of velocity for a difference of 1 foot in height. The value of v varies as $\sqrt{(1 + \alpha t)}$, or approximately as $1 + \frac{1}{2} \alpha t$, t denoting temperature, and α the co-

[1] A more complete discussion of the optics of mirage will be found in two papers by the editor of this work in the *Philosophical Magazine* for March and April, 1873, and in *Nature* for Nov. 19 and 26, 1874.

efficient of expansion, which is $\frac{1}{273}$. Hence if the velocity at 0° denoted by 1, the value at $t°$ will be denoted by $1+\frac{1}{2}\,a\,t$; and if the temperature varies by $\frac{1}{n}$ of a degree per foot, the value of $\frac{\delta v}{v}$ at temperatures near zero will be $\frac{a}{2n}$, that is, $\frac{1}{546n}$, and the radius of curvature will be 546 n feet. This calculation shows that the bending is much more considerable for rays of sound than for rays of light.

1092. Diffraction Fringes.—When a beam of direct sunlight is admitted into a dark room through a narrow slit, a screen placed at any distance to receive it will show a line of white light, bordered with coloured fringes which become wider as the slit is narrowed. They also increase in width as the screen is removed further off. If they are viewed through a piece of red glass which allows only red rays to pass, they will appear as a succession of bands alternately bright and dark.

To explain their origin, we shall suppose the sun's rays (which may be reflected from an external mirror) to be perpendicular to the plane of the slit,[1] so that the wave-fronts are parallel to this plane, and we shall, in the first instance, confine our attention to light of a particular wave-length; for example, that of the light transmitted by the red glass. Then, if the slit be uniform through its whole length, the positions of the bright and dark bands will be governed by the following laws:—

1. The darkest parts will be at points whose distances from the two edges of the slit differ by an exact number of wave-lengths. If the difference be one wave-length, the light which arrives at any instant from different parts of the width of the slit is in all possible phases, and the resultant of the whole is zero. In fact, the disturbance produced by the nearer half of the slit cancels that produced by the remoter half. If the difference be n wave-lengths, we can divide the slit into n parts, such that the effect due to each part is thus *nil.*

2. The brightest parts will be at points whose distances from the two edges of the slit differ by an exact number of wave-lengths *plus*

[1] That is, to the plane of the two knife-edges by which the slit is bounded. This condition can only be strictly fulfilled for a single point on the sun's disc. Every point on the sun's surface sends out its own waves as an independent source; and waves from one point cannot interfere with waves from another. In the experiment as described in the text the fringes due to different parts of the sun's surface are all produced at once on the screen, and overlap each other.

a half. Let the difference be $n+\frac{1}{2}$; then we can divide the slit into n inefficient parts and one efficient part, this latter having only half the width of one of the others.[1]

Each colour of light has its own alternate bands of brightness and darkness, the distance from band to band being greatest for red and least for violet. The superposition of all the bands constitutes the coloured fringes which are seen.

This experiment furnishes the simplest answer to the objection formerly raised to the undulatory theory, that light is not able, like sound, to pass round an obstacle, but can only travel in straight lines. In this experiment light does pass round an obstacle, and turns more and more away from a straight line as the slit is narrowed.

When the slit is not exceedingly narrow, the light sent in oblique directions is quite insensible in comparison with the direct light, and no fringes are visible. "We have reason to think that when *sound* passes through a very large aperture, or when it is reflected from a large surface (which amounts nearly to the same thing), it is hardly sensible except in front of the opening, or in the direction of reflection."[2]

There are several other modes of producing diffraction fringes, which our limits do not permit us to notice. We proceed to describe the mode of obtaining a *pure spectrum* by diffraction.

1093. Diffraction by a Grating.—If a piece of glass is ruled with parallel equidistant scratches (by means of a dividing engine and diamond point) at the rate of some hundreds or thousands to the inch, we shall find, on looking through it at a slit or other bright line (the glass being held so that the scratches are parallel to the slit), that a number of spectra are presented to view, ranged at nearly equal distances, on both sides of the slit. If the experiment is made under favourable circumstances, the spectra will be so pure as to show a number of Fraunhofer's lines.

Instead of viewing the spectra with the naked eye, we may with advantage employ a telescope, focussed on the plane of the slit; or we may project the spectra on a screen, by first placing a convex

[1] Each element of the length of the slit tends to produce a system of circular rings (the screen being supposed parallel to the plane of the slit). If the width of the slit is uniform, these systems will be precisely alike, and will have for their resultant a system of straight bands, parallel to the slit and touching the rings. These are the bands described in the text. Hence, to determine the illumination of any point of the screen, it is only necessary to attend, as in the text, to the nearest points of the two edges of the slit.

[2] Airy, *Undulatory Theory.* Art. 28.

lens so as to form an image of the slit (which must be very strongly illuminated) on the screen, and then interposing the ruled glass in the path of the beam.

A piece of glass thus ruled is called a *grating*.[1] A grating for diffraction experiments consists essentially of a number of parallel strips alternately transparent and opaque.

The distance between the "fixed lines" of the spectra, and the distance from one spectrum to the next, are found to depend on the distance of the strips measured from centre to centre, in other words, on the number of scratches to the inch, but not at all on the relative breadths of the transparent and opaque strips. This latter circumstance only affects the brightness of the spectra.

Diffraction spectra are of great practical importance—

1. As furnishing a uniform standard of reference in the comparison of spectra.

2. As affording the most accurate method of determining the wave-lengths of the different elementary rays of light.

1094. Principle of Diffraction Spectrum.—Let GG (Fig. 774) be a grating, receiving light from an infinitely[2] distant point lying in a direction perpendicular to the plane of the grating, so that the wave-fronts of the incident light are parallel to this plane. Let a convex lens L be placed on the other side of the grating, and let its axis make an acute angle θ with the rays incident on the grating. Then the light collected at its principal focus F consists of all the light incident upon the lens parallel to its axis. Let s denote the distance

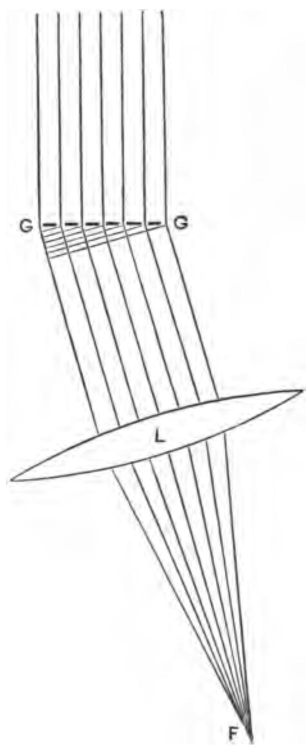


Fig. 774.
Principle of Diffraction Spectrum.

[1] Engraved glass gratings of sufficient size for spectroscopic purposes (say an inch square) are extremely expensive and difficult to procure. Lord Rayleigh has made numerous photographic copies of such gratings, and the copies appear to be equally effective with the originals.

[2] It is not *necessary* that the source should be infinitely distant (or the incident rays parallel); but this is the simplest case, and the most usual case in practice.

between the rulings, measured from centre to centre, so that if, for example, there are 1000 lines to the inch, s will be $\frac{1}{1000}$ of an inch; and suppose first that $s \sin \theta$ is exactly equal to the wave-length λ of one of the elementary kinds of light. Then, of all the light which falls upon the lens parallel to its axis, the left-hand portion in the figure is most retarded (having travelled farthest), and the right-hand portion least, the retardation, in comparing each transparent interval with the next, being constant, and equal to $s \sin \theta$, as is evident from an inspection of the figure. Now, for the particular kind of light for which $\lambda = s \sin \theta$, this retardation is exactly a wave-length, and all the transparent intervals send light of the same phase to the focus F; so that, if there are 1000 such intervals, the resultant amplitude of vibration at F is 1000 times the amplitude due to one interval alone. For light of any other wave-length this coincidence of phase will not exist. For example, if the difference between λ and $s \sin \theta$ is $\frac{1}{1000} \lambda$, the difference of phase between the lights received from the 1st and 2d intervals will be $\frac{1}{1000} \lambda$, between the 1st and 3d $\frac{2}{1000} \lambda$, between the 1st and 501st $\frac{500}{1000} \lambda$, or just half a wave-length, and so on. The 1st and 501st are thus in complete discordance, as are also the 2d and 502d, &c. Light of every wave-length except one is thus almost completely destroyed by interference, and the light collected at F consists almost entirely of the particular kind defined by the condition

$$\lambda = s \sin \theta. \qquad (1)$$

The purity of the diffraction spectrum is thus explained.

If a screen be held at F, with its plane perpendicular to the principal axis, any point on this screen a little to one side of F will receive light of another definite wave-length, corresponding to another direction of incidence on the lens, and a pure spectrum will thus be depicted on the screen.

1095. Practical Application.—In the arrangement actually employed for accurate observation, the lens L L is the object-glass of a telescope with a cross of spider-lines at its principal focus F. The telescope is first pointed directly towards the source of light, and is then turned to one side through a measured angle θ. Any fixed line of the spectrum can thus be brought into apparent coincidence with the cross of spider-lines, and its wave-length can be computed by the formula (1).

The spectrum to which formula (1) relates is called the *spectrum of the first order.*

There is also a spectrum of the second order, corresponding to values of θ nearly twice as great, and for which the equation is

$$2\lambda = s \sin\theta. \qquad (2)$$

For the spectrum of the third order, the equation is

$$3\lambda = s \sin\theta; \qquad (3)$$

and so on, the explanation of their formation being almost precisely the same as that above given. There are two spectra of each order, one to the right, and the other at the same distance to the left of the direction of the source. In Ångstrom's observations,[1] which are the best yet taken, all the spectra, up to the sixth inclusive, were observed, and numerous independent determinations of wave-length were thus obtained for several hundred of the dark lines of the solar spectrum.

The source of light was the infinitely distant image of an illuminated slit, the slit being placed at the principal focus of a collimator, and illuminated by a beam of the sun's rays reflected from a mirror.

The purity of a diffraction spectrum increases with the number of lines on the grating which come into play, provided that they are exactly equidistant; and may therefore be increased either by increasing the size of the grating, or by ruling its lines closer together. The gratings employed by Ångstrom were about $\frac{3}{4}$ of an inch square, the closest ruled having about 4500 lines, and the widest 1500.

As regards brightness, diffraction spectra are far inferior to those obtained by prisms. To give a maximum of light, the opaque intervals should be perfectly opaque, and the transparent intervals perfectly transparent; but even under the most favourable conditions, the whole light of any one of the spectra cannot exceed about $\frac{1}{10}$ of the light which would be received by directing the telescope to the slit. The greatest attainable intrinsic brightness in any part of a diffraction spectrum is thus not more than $\frac{1}{10}$ of the intrinsic brightness in the same part of a prismatic spectrum, obtained with the same slit, collimator, and observing telescope, and with the same angular separation of fixed lines. The brightness of the spectra partly depends upon the ratio of the breadths of the transparent and opaque intervals. In the case of the spectra of the first order, the best ratio is that of equality, and equal departures from equality in opposite directions give identical results; for example, if the breadth

[1] Ångstrom, *Recherches sur la Spectre solaire.* Upsal, 1868.

of the transparent intervals is to the breadth of the opaque either as 1 : 5 or as 5 : 1, it can be shown that the quantity of light in the first spectrum is just a quarter of what it would be with the breadths equal.

When a diffraction spectrum is seen with the naked eye, the cornea and crystalline of the eye take the place of the lens L L, and form a real image on the retina at F.

1096. Retardation Gratings.—If, instead of supposing the bars of the grating to be opaque, we suppose them to be transparent, but to produce a definite change of phase either by acceleration or retardation, the spectra produced will be the same as in the case above discussed, except as regards brightness. We may regard the effect as consisting of the superposition of two exactly coincident sets of spectra, one due to the spaces and the other to the bars. Any one of the resultant spectra may be either brighter or less bright than either of its components, according to the difference of phase between them. If the bars and spaces are equally transparent, the two superimposed spectra will be equally bright, and their resultant at any part may have any brightness intermediate between zero and four times that of either component.

1097. Reflection Gratings.—Diffraction spectra can also be obtained by reflection from a surface of speculum metal finely ruled with parallel and equidistant scratches. The appearance presented is the same as if the geometrical image of the slit (with respect to the grating regarded as a plane mirror) were viewed through the grating regarded as transparent.

1098. Standard Spectrum.—The simplicity of the law connecting wave-length with position, in the spectra obtained by diffraction, offers a remarkable contrast to the "irrationality" of the dispersion produced by prisms. Diffraction spectra may thus be fairly regarded as natural standards of comparison; and, in particular, the limiting form (if we may so call it) to which the diffraction spectra tend, as sin θ becomes small enough to be identified with θ, so that deviation becomes simply proportional to wave-length, is generally and deservedly accepted by spectroscopists as the *absolute standard of reference.* This limiting form is often briefly designated as "the diffraction spectrum;" it differs in fact to a scarcely appreciable extent from the first, or even the second and third spectra furnished in ordinary cases by a grating.

The diffraction spectrum differs notably from prismatic spectra in

the much greater relative extension of the red end. Owing to this circumstance, the brightest part of the diffraction spectrum of solar light is nearly in its centre.

The first three columns of numbers in the subjoined table indicate the approximate distances between the fixed lines B, D, E, F, G in certain prismatic spectra, and in the standard diffraction spectrum, the distance from B to G being in each case taken as 1000:—

	Flint-glass. Angle of 60°.	Bisulphide of Carbon. Angle of 60°.	Diffraction, or Difference of Wave-length.	Difference of Wave-frequency.
B to D, . .	220	194	381	278
D to E, . .	214	206	243	232
E to F, . .	192	190	160	184
F to G, . .	374	410	216	306
	1000	1000	1000	1000

In the standard diffraction spectrum, deviation is simply proportional to wave-length, and therefore the distance between two colours represents the difference of their wave-lengths. It has been suggested that a more convenient reference-spectrum would be constructed by assigning to each colour a deviation proportional to its wave-frequency (or to the reciprocal of its wave-length), so that the distance between two colours will represent the difference between their wave-frequencies. The result of thus disposing the fixed lines is shown in the last column of the above table. It differs from prismatic spectra in the same direction, but to a much less extent than the diffraction spectrum.

It has been suggested by Mr. Stoney as extremely probable, that the bright lines of spectra are in many cases harmonics of some one fundamental vibration. Three of the four bright lines of hydrogen have wave-frequencies exactly proportional to the numbers 20, 27, and 32; and in the spectrum of chloro-chromic acid all the lines whose positions have been observed (31 in number) have wave-frequencies which are multiples of one common fundamental.

1099. Wave-lengths.—Wave-lengths of light are commonly stated in terms of a unit of which 10^{10} make a metre,—hence called the *tenth-metre*. The following are the wave-lengths of some of the principal "fixed lines" as determined by Ångstrom:[1]—

[1] The wave-lengths of the spectral lines of all elementary substances will be found in Dr. W. M. Watts' *Index of Spectra;* and the wave-lengths and wave-frequencies of the dark lines in the solar spectrum, with the names of the substances to which many of them are due, will be found in the *British Association Report* for 1878 (Dublin), pp. 40–91.

WAVE-LENGTHS IN TENTH-METRES.

A	7604	E	5269
B	6867	F	4861
C	6562	G	4307
D_2	5895	H_1	3968
D_1	5889	H_2	3933

The velocity of light is 300 million metres per second, or 300×10^{16} tenth-metres per second. The number of waves per second for any colour is therefore 300×10^{16} divided by its wave-length as above expressed. Hence we find approximately:—

For A	395	millions	of millions	per second.
„ D	510	„	„	„
„ H	760	„	„	„

1100. Colours of Thin Films. Newton's Rings.—If two pieces of glass, with their surfaces clean, are brought into close contact, coloured fringes are seen surrounding the point where the contact is closest. They are best seen when light is obliquely reflected to the eye from the surfaces of the glass, and fringes of the complementary colours may be seen by transmitted light. A drop of oil placed on the surface of clean water spreads out into a thin film, which exhibits similar fringes of colour; and in general, a very thin film of any transparent substance, separating media whose indices of refraction are different from its own, exhibits colour, especially when viewed by obliquely reflected light. In the first experiment above-mentioned, the thin film is an air-film separating the pieces of glass. In soap-bubbles or films of soapy water stretched on rings, a similar effect is produced by a small thickness of water separating two portions of air.

The colours, in all these cases, when seen by reflected light, are produced by the mutual interference of the light reflected from the two surfaces of the thin film. An incident ray undergoes, as explained in § 992, a series of reflections and refractions; and we may thus distinguish, for light of any given refrangibility, several systems of waves, all of which originally came from the same source. These systems give by their interference a series of alternately bright and dark fringes; and when ordinary white light is employed, the fringes are broadest for the colours of greatest wave-length. Their superposition thus produces the observed colours. The colours seen by transmitted light may be similarly explained.

The first careful observations of these coloured fringes were made by Newton, and they are generally known as *Newton's rings.*

1100 A. Concave Gratings.—Great progress has been made in recent years in the manufacture of reflection gratings ruled on speculum metal. Mr. Rutherfurd of New York has produced several specimens (containing thirty thousand lines in a space of about an inch and three quarters), which give better results than have ever been obtained by means of prisms. Professor Rowland of Baltimore has improved upon these, and constructed gratings with 160,000 lines in a space of about six inches.

Professor Rowland has also introduced the novelty of ruling gratings on concave spherical surfaces, all previous gratings having been ruled on plane surfaces. He is thus enabled to dispense both with a collimating lens and with the object-glass of the observing telescope. As this invention promises to be very important, we shall explain it at some length.

The rule which determines the kind of light that will be thrown in a given direction by any part of a reflection grating is, that the total length of path from the slit to a point lying in this direction differs by one wave-length or an exact number of wave-lengths of this particular light, as we pass from one bar to the next. If the slit is at the same distance from all the bars, the difference of path will be the difference of the two reflected rays, and will be the projection of the distance between the bars on one of these rays; so that, as in §1094, we shall have $s \sin \theta$ equal to λ or a multiple of λ, θ now denoting the angle of reflection.

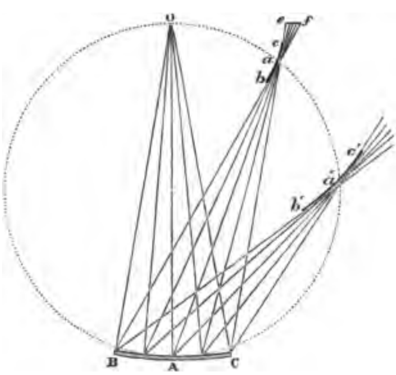


Fig. 774A.—Concave Grating.

In Fig. 774A, O is the centre of the sphere of which the grating forms part, A the middle point of the grating, B A C a section of the grating perpendicular to the rulings.

First let the slit be at O. For the spectra of the first order the angle of reflection θ for a given kind of light is determined by the equation

$$s \sin \theta = \lambda,$$

and if, round O as centre, we describe a circle of radius O A $\sin \theta$, this circle will be touched by all the reflected rays. An arc $b\,a\,c$

of this circle containing the same number of degrees as the grating B A C will be the caustic. The narrowest part of the sheaf of rays will be at a the middle point of this arc, being the point where it is touched by the ray A a from the middle point of the grating. Since O a if joined is at right angles to A a, the point a lies on the circle described on O A as diameter—the dotted circle in the figure. For the spectra of the second order, θ is determined by the equation

$$s \sin \theta = 2 \lambda,$$

and the caustic will be the circular arc $b'a'c'$ described about O as centre with a radius double of the radius of $b\,a\,c$, the construction being in other respects the same; and similar reasoning applies to the spectra of higher orders.

All the caustics of every order and for every value of λ will thus have their brightest points on the circle described on O A as diameter.

If the slit be at a, the reflected rays of the first order will form one caustic at O and another at a'; for the rays incident from a on consecutive bars of the grating at A will have a common difference of λ, and the distances of these bars from a' have a common difference of 2 λ.

Neglecting the breadth of the reflected beam at its narrowest part, we may regard a and a' as foci conjugate to O, and may regard the dotted circle as the locus of a point from which rays to all parts of the arc B A C have the same angle of incidence; whence it follows that, if the slit be anywhere on this circle, all the spectra will focus themselves along the circle. Hence the grating, the slit, and the screen for receiving the spectra, or the eye-piece for viewing them, may be fixed at the ends of three equal arms all pivoted at the centre of this circle.

A pencil of rays from a single point at O will not meet *in a single point* at a, even if we regard the breadth of the beam at a as negligible, but in a focal line perpendicular to the plane of the diagram, and they will meet again in a second focal line $e\,f$ in the plane of the diagram. The lines of the spectrum due to a slit at O, perpendicular to the plane of the diagram, will therefore be focussed at a, while the transverse lines due to particles of dust in the slit will be focussed at $e\,f$. Similar remarks apply to the spectra of the second and higher orders. Hence the spectra as actually observed with a concave grating have the advantage of not showing dust lines.

CHAPTER LXXV.

POLARIZATION AND DOUBLE REFRACTION.

1101. Polarization.—When a piece of the semi-transparent mineral called tourmaline is cut into slices by sections parallel to its axis, it is found that two of these slices, if laid one upon the other in a particular relative position, as A, B (Fig. 775), form an opaque combination. Let one of them, in fact, be turned round upon the other through various angles (Fig. 775). It will be found that the combination is most transparent in two positions differing by 180°, one of them *a b* being the natural position which they originally occupied in the crystal; and that it is most opaque in the two positions at right angles to these. It is not necessary that the slices should be cut from the same crystal. Any two plates of tourmaline with their faces parallel to the axis of the crystals from which they were cut, will exhibit the same phenomenon. The experiment shows that light which has passed through one such plate is in a peculiar and so to speak unsymmetrical condition. It is said to be *plane-polarized.* According to the undulatory theory, a ray of common light contains vibrations in all planes passing through the ray, and a ray of plane-polarized light contains vibrations in one plane only. Polarized light cannot be distinguished from common light by the naked eye; and for all experiments in polarization two pieces of apparatus must be employed —one to produce polarization, and the other to show it. The former is called the *polarizer,* the latter the *analyser;* and every apparatus that serves for one of these purposes will also serve for the other. In the experiment above described, the plate next the eye is the

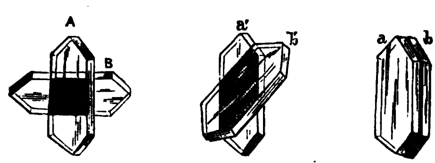


Fig. 775.—Tourmaline Plates.

analyser. The usual process in examining light with a view to test whether it is polarized, consists in looking at it through an analyser, and observing whether any change of brightness occurs as the analyser is rotated. When the light of the blue sky is thus examined, a difference of brightness can always be detected according to the position of the analyser, especially at the distance of about 90° from the sun. In all such cases there are two positions differing by 180°, which give a minimum of light, and the two positions intermediate between these give a maximum of light.

The extent of the changes thus observed is a measure of the completeness of the polarization of the light.

1102. Polarization by Reflection.—Transmission through tourmaline is only one of several ways in which light can be polarized. When a beam of light is reflected from a polished surface of glass, wood, ivory, leather, or any other non-metallic substance, at an angle of from 50° to 60° with the normal, it is more or less polarized, and in like manner a reflector composed of any of these substances may be employed as an analyser. In so using it, it should be rotated about an axis parallel to the incident rays which are to be tested, and the observation consists in noting whether this rotation produces changes in the amount of reflected light.

Malus' Polariscope (Fig. 776) consists of two reflectors A, B, one serving as polarizer and the other as analyser, each consisting of a pile of glass plates. Each of these reflectors can be turned about a horizontal axis; and the upper one (which is the analyser) can also be turned about a vertical axis, the amount of rotation being measured on the horizontal circle C C. To obtain the most powerful effects, each of the reflectors should be set at an angle of about 33° to the vertical, and a strong beam of common light should be allowed to fall upon the lower pile in such a direction as to be reflected vertically upwards. It will thus fall upon the centre of the upper pile, and the angles of incidence and reflection on both the piles will be about 57°. The observer looking into the upper pile, in such a

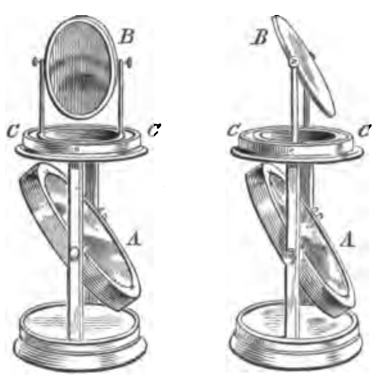


Fig. 776.—Malus' Polariscope.

direction as to receive the reflected beam, will find that, as the upper pile is rotated about a vertical axis, there are two positions (differing by 180°) in which he sees a black spot in the centre of the field of view, these being the positions in which the upper pile refuses to reflect the light reflected to it from the lower pile. They are 90° on either side of the position in which the two piles are parallel; this latter, and the position differing from it by 180°, being those which give a maximum of reflected light.

For every reflecting substance there is a particular angle of incidence which gives a maximum of polarization in the reflected light. It is called the *polarizing angle* for the substance, and its tangent is always equal to the index of refraction of the substance; or what amounts to the same thing, it is that particular angle of incidence which is the complement of the angle of refraction, so that the refracted and reflected rays are at right angles.[1] This important law was discovered experimentally by Sir David Brewster.

The reflected ray under these circumstances is in a state of almost complete polarization; and the advantage of employing a *pile* of plates consists merely in the greater intensity of the reflected light thus furnished. The transmitted light is also polarized; it diminishes in intensity, but becomes more completely polarized, as the number of plates is increased. The reflected and the transmitted light are in fact mutually complementary, being the two parts into which common light has been decomposed; and their polarizations are accordingly opposite, so that, if both the transmitted and reflected beams are examined by a tourmaline, the maxima of obscuration will be obtained by placing the axis of the tourmaline in the one case parallel and in the other perpendicular to the plane of incidence.

It is to be noted that what is lost in reflection is gained in transmission, and that polarization never favours reflection at the expense of transmission.

1103. Plane of Polarization.—That particular plane in which a ray of polarized light, incident at the polarizing angle, is most copiously reflected, is called the *plane of polarization* of the ray. When the polarization is produced by reflection, the plane of reflection is the

[1] Adopting the indices of refraction given in the table § 986, we find the following values for the polarizing angle for the undermentioned substances:—

Diamond, . . .	67° 43′ to 70° 3′	Crown-glass, . .	56° 51′ to 57° 23′
Flint-glass, . .	57° 36′ to 58° 40′	Pure Water, ...	53° 11′

plane of polarization. According to Fresnel's theory, which is that generally received, the vibrations of light polarized in any plane are perpendicular to that plane (§ 1115). The vibrations of a ray reflected at the polarizing angle are accordingly to be regarded as perpendicular to the plane of incidence and reflection, and therefore as parallel to the reflecting surface.

1104. Polarization by Double Refraction.—We have described in § 998 some of the principal phenomena of double refraction in uniaxal crystals. We have now to mention the important fact that the two rays furnished by double refraction are polarized, the polarization in this case being more complete than in any of the cases thus far discussed. On looking at the two images through a plate of tourmaline, or any other analyser, it will be found that they undergo great variations of brightness as the analyser is rotated, one of them becoming fainter whenever the other becomes brighter, and the maximum brightness of either being simultaneous with the absolute extinction of the other. If a second piece of Iceland-spar be used as the analyser, four images will be seen, of which one pair become dimmer as the other pair become brighter, and either of these pairs can be extinguished by giving the analyser a proper position.

1105. Theory of Double Refraction.—The existence of double refraction admits of a very natural explanation on the undulatory theory. In uniaxal crystals it is assumed that the elasticity of the luminiferous æther is the same for all vibrations executed in directions perpendicular to the axis; and that, for vibrations in other directions, the elasticity varies solely according to the inclination of the direction of vibration to the axis. There are two classes of doubly-refracting uniaxal crystals, called respectively *positive* and *negative*. In the former the elasticity for vibrations perpendicular to the axis is a maximum; in the latter it is a minimum. Iceland-spar belongs to the latter class; and as small elasticity implies slow propagation, a ray propagated by vibrations perpendicular to the axis will, in this crystal, travel with minimum velocity; while the most rapid propagation will be attained by rays whose vibrations are parallel to the axis.

Consider any plane oblique to the axis. Through any point in this plane we can draw one line perpendicular to the axis; and the line at right angles to this will have smaller inclination to the axis than any other line in the plane. These two lines are the directions of least and greatest resistance to vibration; the former is the direc-

tion of vibration for an ordinary, and the latter for an extraordinary ray. The velocity of propagation is the same for the ordinary rays in all directions in the crystal, so that the wave-surface for these is spherical; but the velocity of propagation for the extraordinary rays differs according to their inclination to the axis, and their wave-surface is a spheroid whose polar diameter is equal to the diameter of the aforesaid sphere. The sphere and spheroid touch one another at the extremities of this diameter (which is parallel to the axis of the crystal), and the ordinary and extraordinary rays coincide both in direction and velocity along this common diameter. The general construction for the path of the extraordinary ray is due to Huygens, and has been described in § 1081, Fig. 770, where C A is the incident and A F the refracted ray.

When the plane of incidence contains the axis, the spheroid will be symmetrical with respect to this plane; and, therefore, when we draw the tangent plane E F perpendicular to the plane of incidence (as directed in the construction) the point of contact F will lie in the plane of incidence.

Another special case is that in which the plane of incidence is perpendicular to the axis, and the refracting surface parallel to the axis. In this case also the spheroid will be symmetrical with respect to the plane of incidence, which will in fact be the equatorial plane of the spheroid, and the point of contact F will as before lie in this plane. Moreover since the section is equatorial it is a circle, and hence, as shown in Fig. 771, the law of sines will be applicable. The ratio of the sines of the angles of incidence and refraction for this particular case is called the *extraordinary index* of refraction for the crystal. It is the ratio of the velocity in air to the velocity along an equatorial radius of the spheroid.

In general, the spheroid is not symmetrical with respect to the plane of incidence, and the refracted ray A F does not lie in this plane.

Tourmaline, like Iceland-spar, is a negative uniaxal crystal; and its use as a polarizer depends on the property which it possesses of absorbing the ordinary much more rapidly than the extraordinary ray, so that a thickness which is tolerably transparent to the latter is almost completely opaque to the former.

1106. Nicol's Prism.—One of the most convenient and effective contrivances for polarizing light, or analysing it when polarized, is that known, from the name of its inventor, as Nicol's prism. It is

made by slitting a rhomb of Iceland-spar along a diagonal plane *a c b d* (Fig. 777), and cementing the two pieces together in their natural position by Canada balsam, a substance whose refractive index is intermediate between the ordinary and extraordinary indices of the crystal.[1] A ray of common light S I undergoes double refraction on entering the prism. Of the two rays thus formed, the ordinary ray is totally reflected on meeting the first surface of the balsam, and passes out at one side of the crystal, as *o* O; while the extraordinary ray is transmitted through the balsam as through a parallel plate, and finally emerges at the end of the prism, in the direction *e* E, parallel to the original direction S I. This apparatus has nearly all the convenience of a tourmaline plate, with the advantages of much greater transparency and of complete polarization.

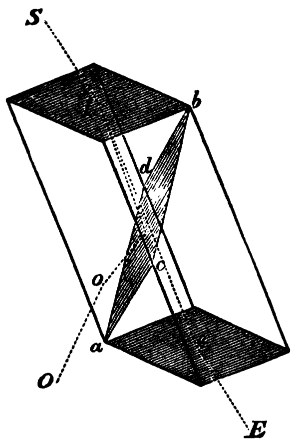


Fig. 777.—Nicol's Prism.

In Foucault's prism, which is extensively used instead of Nicol's, the Canada balsam is omitted, and there is nothing but air between the two pieces. This change has the advantage of shortening the prism (because the critical angle of total reflection depends on the relative index of refraction of the two media), but gives a smaller field of view, and rather more loss of light by reflection.

1107. Colours produced by Elliptic Polarization.—Very beautiful colours may be produced by the peculiar action of polarized light. For example, if a piece of selenite (crystallized gypsum) about the thickness of paper, is introduced between the polarizer and analyser of any polarizing arrangement, and turned about into different directions, it will in some positions appear brightly coloured, the colour being most decided when the analyser is in either of the two critical positions which give respectively the greatest light and the greatest darkness. The colour is changed to its complementary by

[1] *a* and *b* are the corners at which three equal obtuse angles meet (§ 999). The ends of the rhomb which are shaded in the figure are rhombuses. Their diagonals drawn through *a* and *b* respectively will lie in one plane, which will contain the axis of the crystal, and will cut the plane of section *a c b d* at right angles. The length of the rhomb is about three and a half times its breadth.

rotating the analyser through a right angle; but rotation of the piece of selenite, when the analyser is in either of the critical positions, merely alters the depth of the colour without changing its tint, and in certain critical positions of the selenite there is a complete absence of colour. Thicker plates of selenite restore the light when extinguished by the analyser, but do not show colour.

1108. Explanation.—The following is the explanation of these appearances. Let the analyser be turned into such a position as to produce complete extinction of the plane-polarized light which comes to it from the polarizer; and let the plane of polarization and the plane perpendicular thereto (and parallel to the polarized rays) be called the two *planes of reference.* Let the slice of selenite be laid so that the polarized rays pass through it normally. Then there are two directions, at right angles to each other, which are the directions of greatest and least elasticity in the plane of the slice. Unless the slice is laid so that these directions coincide with the two planes of reference, the plane-polarized light which is incident upon it will be broken up into two rays, one of which will traverse it more rapidly than the other. Referring to the diagram of Lissajous' figures (Fig. 634), let the sides of the rectangle be the directions of greatest and least elasticity, and let the diagonal line in the first figure be the direction of the vibrations of an incident ray,—this diagonal accordingly lies in one of the two planes of reference. In traversing the slice, the component vibrations in the directions of greatest and least elasticity will be propagated with unequal velocities; and if the incident ray be homogeneous, the emergent light will be elliptically polarized; that is to say, its vibrations, instead of being rectilinear, will be elliptic, precisely on the principle[1] of Blackburn's pendulum (§ 924). The shape of the ellipse depends, as in the case of Lissajous' figures, on the amount of retardation of one of the two component vibrations as compared with the other, and this is directly proportional to the thickness of the slice. The analyser resolves these elliptic vibrations into two rectilinear components parallel and perpendicular to the original direction of vibration, and suppresses one of these components, so that only the other remains.

[1] The principle is that, whereas displacement of a particle parallel to either of the sides of the rectangle calls out a restoring force directly opposite to the displacement, displacement in any other direction calls out a restoring force inclined to the direction of displacement, being in fact the resultant of the two restoring forces which its two components parallel to the sides of the rectangle would call out.

Thus if the ellipse in the annexed figure (Fig. 778) represent the vibrations of the light as it emerges from the selenite, and CD, E F be tangents parallel to the original direction of vibration, the perpendicular distance between these tangents, A B, is the component vibration which is not suppressed when the analyser is so turned that all the light would be suppressed if the selenite were removed. By rotating the analyser, we shall obtain vibrations of various amplitudes, corresponding to the distances between parallel tangents drawn in various directions.

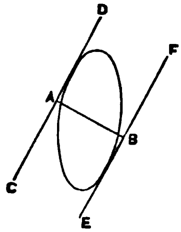


Fig. 778.—Colours of Selenite Plates.

For a certain thickness of selenite the ellipse may become a circle, and we have thus what is called *circularly polarized* light, which is characterized by the property that rotation of the analyser produces no change of intensity. Circularly polarized light is not however identical with ordinary light; for the interposition of an additional thickness of selenite converts it into elliptically (or in a particular case into plane) polarized light (§ 1114).

The above explanation applies to homogeneous light. When the incident light is of various refrangibilities, the retardation of one component upon the other is greatest for the rays of shortest wavelength. The ellipses are accordingly different for the different elementary colours, and the analyser in any given position will produce unequal suppression of different colours. But since the component which is suppressed in any one position of the analyser, is the component which is not suppressed when the analyser is turned through a right angle, the light yielded in the former case *plus* the light yielded in the latter must be equal to the whole light which was incident on the selenite.[1] Hence the colours exhibited in these two positions must be complementary.

It is necessary for the exhibition of colour in these experiments that the plate of selenite should be very thin, otherwise the retardation of one component vibration as compared with the other will be greater by several complete periods for violet than for red, so that the ellipses will be identical for several different colours, and the total non-suppressed light will be sensibly white in all positions of the analyser.

[1] We here neglect the light absorbed and scattered; but the loss of this does not sensibly affect the *colour* of the whole. It is to be borne in mind that the intensity of light is measured by the *square* of the amplitude, and is therefore the simple sum of the intensities of its two components when the resolution is rectangular.

Two thick plates may however be so combined as to produce the effect of one thin plate. For example, two selenite plates, of nearly equal thickness, may be laid one upon the other, so that the direction of greatest elasticity in the one shall be parallel to that of least elasticity in the other. The resultant effect in this case will be that due to the difference of their thicknesses. Two plates so laid are said to be *crossed.*

1109. Colours of Plates perpendicular to Axis.—A different class of

Fig. 779.—Rings and Cross.

appearances are presented when a plate, cut from a uniaxal crystal by sections perpendicular to the axis, is inserted between the polarizer and the analyser. Instead of a broad sheet of uniform colour, we have now a system of coloured rings, interrupted, when the analyser is in one of the two critical positions, by a black or white cross, as at A, B (Fig. 779).

1110. Explanation.—The following is the explanation of these appearances. Suppose, for simplicity, that the analyser is a plate of tourmaline held close to the eye. Then the light which comes to the eye from the nearest point of the plate under examination (the foot of a perpendicular dropped upon it from the eye), has traversed the plate normally, and therefore parallel to its optic axis. It has therefore not been resolved into an ordinary and an extraordinary ray, but has emerged from the plate in the same condition in which it entered, and is therefore black, gray, or white according to the position of the analyser, just as it would be if the plate were removed. But the light which comes obliquely to the eye from any other part of the plate, has traversed the plate obliquely, and has undergone double refraction. Let E (Fig. 780) be the position

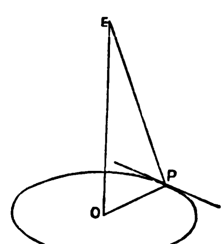


Fig. 780.
Theory of Rings and Cross.

of the eye, E O a perpendicular on the plate, P a point on the circumference of a circle described about O as centre. Then, since E O is parallel to the axis of the plate, the direction of vibration for the ordinary ray at P is perpendicular to the plane E O P, and is tangential to the circle. The direction of vibration for the extraordinary ray lies in the plane E O P, is nearly perpendicular to E O (or to the axis), if the angle O E P is small, and deviates more from perpendicularity to the axis as the angle O E P increases. Both for this reason, and also on account of the greater thickness traversed, the retardation of one ray upon the other is greater as P is taken further from O; and from the symmetry of the circumstances, it must be the same at the same distance from O all round. In consequence of this retardation, the light which emerges at P in the direction PE is elliptically polarized; and by the agency of the analyser it is accordingly resolved into two components, one of which is suppressed. With homogeneous light, rings alternately dark and bright would thus be formed at distances from O corresponding to retardations of 0, ½, 1, 1½, 2, 2½, . . . complete periods; and it can be shown that the radii of these rings would be proportional to the numbers 0, $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, $\sqrt{6}$: . . . The rings are larger for light of long than of short wave-length; and the coloured rings actually exhibited when white light is employed, are produced by the superposition of all the systems of monochromatic rings. The monochromatic rings for red light are easily seen by looking at the actual rings through a piece of red glass.

Let O, P, Fig. 781, be the same points which were denoted by these letters in Fig. 780, and let A B be the direction of vibration of the light incident on the crystal at P. Draw A C, D B parallel to O P, and complete the rectangle A C B D. Then the length and breadth of this rectangle are approximately the directions of vibration of the two components, one of which loses upon the other in traversing the crystal. The vibration of the emergent ray is represented by an ellipse inscribed in the rectangle A C B D (§ 922, note 2); and when the loss is half a period, this ellipse shrinks into a straight line, namely, the diagonal C D. Through C and D draw lines parallel to A B; then the distance between these parallels

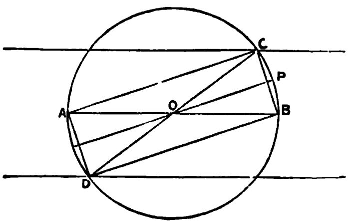


Fig. 781.—Theory of Rings and Cross.

represents the double amplitude of the vibration which is transmitted when there has been a retardation of half a period, and is greater than the distance between the tangents in the same direction to any of the inscribed ellipses. A retardation of another half period will again reduce the inscribed ellipse to the straight line A B, as at first. The position D C corresponds to the brightest and A B to the darkest part of any one of the series of rings for a given wave-length of light, the analyser being in the position for suppressing all the light if the crystal were removed. When the analyser is turned into the position at right angles to this, A B corresponds to the brightest, and D C to the darkest parts of the rings. It is to be remembered that amount of retardation depends upon distance from the centre of the rings, and is the same all round. The two diagonals of our rectangle therefore correspond to different sizes of rings.

If the analyser is in such a position with respect to the point P considered, that the suppressed vibration is parallel to one of the sides of the rectangle (in other words, if O P, or a line perpendicular to O P, is the direction of suppression) the retardation of one component upon the other has no influence, inasmuch as one of the two components is completely suppressed and the other is completely transmitted. There are, accordingly, in all positions of the analyser, a pair of diameters, coinciding with the directions of suppression and non-suppression, which are alike along their whole length and free from colour.

Again if P is situated at B or at 90° from B, the corner C of the rectangle coincides with B or with A, and the rectangle, with all its inscribed ellipses, shrinks into the straight line A B. The two diameters coincident with and perpendicular to A B are therefore alike along their whole length and uncoloured.

The two colourless crosses which we have thus accounted for, one of them turning with the analyser and the other remaining fixed with the polarizer, are easily observed when the analyser is not near the critical positions. In the critical positions, the two crosses come into coincidence; and these are also the positions of maximum blackness or maximum whiteness for the two crosses considered separately. Hence the conspicuous character of the cross in either of these positions, as represented at A, B, Fig. 779. As the analyser is turned away from these positions, the cross at first turns after it with half its angular velocity, but soon breaks up into rings, some-

what in the manner represented at C, which corresponds to a position not differing much from A.

1111. **Biaxal Crystals.**—Crystals may be divided optically into three classes:—

1. Those in which there is no distinction of different directions, as regards optical properties. Such crystals are said to be optically *isotropic*.

2. Those in which the optical properties are the same for all directions equally inclined to one particular direction called the optic axis, but vary according to this inclination. Such crystals are called *uniaxal*.

3. All remaining crystals (excluding compound and irregular formations) belong to the class called *biaxal*. In any homogeneous elastic solid, there are three cardinal directions called *axes of elasticity*, possessing the same distinctive properties which belong to the two principal planes of vibration in Blackburn's pendulum (§ 924); that is to say, if any small portion of the solid be distorted by forcibly displacing one of its particles in one of these cardinal directions, the forces of elasticity thus evoked tend to urge the particle *directly* back; whereas displacement in any other direction calls out forces whose resultant is generally oblique to the direction of displacement, so that when the particle is released it does not fly back through the position of equilibrium, but passes on one side of it, just as the bob of Blackburn's pendulum generally passes beside and not through the lowest point which it can reach.

In biaxal crystals, the resistances to displacement in the three cardinal directions are all unequal; and this is true not only for the crystalline substance itself, but also for the luminiferous æther which pervades it, and is influenced by it.[1] The construction given by Fresnel for the wave-surface in any crystal is as follows:—First take an ellipsoid, having its axis parallel to the three cardinal directions, and of lengths depending on the particular crystalline substance considered. Then let any plane sections (which will of course be ellipses) be made through the centre of this ellipsoid, let normals to them be drawn through the centre, and on each normal let points be taken at distances from the centre equal to the greatest and least radii of the corresponding section. The locus of these points is the complete wave-surface, which consists of two sheets cutting one

[1] The cardinal directions are however believed not to be the same for the æther as for the material of the crystal.

another at four points. These four points of intersection are situated upon the normals to the two *circular sections* of the ellipsoid, and the two *optic axes*, from which *biaxal* crystals derive their name, are closely related to these two circular sections. The optic axes are the directions of *single wave-velocity*, and the normals to the two circular sections are the directions of *single ray-velocity*. The direction of advance of a wave is always regarded as normal to the front of the wave, whereas the direction of a ray (defined by the condition of traversing two apertures placed in its path) always passes through the centre of the wave-surface, and is not in general normal to the front. Both these pairs of directions of single velocity are in the plane which contains the greatest and least axes of the ellipsoid.

When two axes of the ellipsoid are equal, it becomes a spheroid, and the crystal is uniaxal. When all three axes are equal, it becomes a sphere, and the crystal is isotropic.

Experiment has shown that biaxal crystals expand with heat unequally in three cardinal directions, so that in fact a spherical piece of such a crystal is changed into an ellipsoid[1] when its temperature is raised or lowered. A spherical piece of a uniaxal crystal in the same circumstances changes into a spheroid; and a spherical piece of an isotropic crystal remains a sphere.

It is generally possible to determine to which of the three classes a crystal belongs, from a mere inspection of its shape as it occurs in nature. Isotropic crystals are sometimes said to be symmetrical about a point, uniaxal crystals about a line, biaxal crystals about neither. The following statement is rather more precise:—

If there is one and only one line about which if the crystal be rotated through 90° or else through 120° the crystalline form remains in its original position, the crystal is uniaxal, having that line for the axis. If there is more than one such line, the crystal is isotropic, while, if there is no such line, it is biaxal. Even in the last case, if there exist a plane of crystalline symmetry, such that one half of the crystal is the reflected image of the other half with respect to this plane, it is also a plane of optical symmetry, and one of the three cardinal directions for the æther is perpendicular to it.[2]

[1] This fact furnishes the best possible definition of an ellipsoid for persons unacquainted with solid geometry.

[2] The optic axes either lie in the plane of symmetry, or lie in a perpendicular plane and are equally inclined to the plane of symmetry.

For the precise statement here given, the Editor is indebted to Professor Stokes.

Glass, when in a strained condition, ceases to be isotropic, and if inserted between a polarizer and an analyser, exhibits coloured streaks or spots, which afford an indication of the distribution of strain through its substance. The experiment is shown sometimes with unannealed glass, which is in a condition of permanent strain, sometimes with a piece of ordinary glass which can be subjected to force at pleasure by turning a screw. Any very small portion of a piece of strained glass has the optical properties of a crystal, but different portions have different properties, and hence the glass as a whole does not behave like one crystal.

The production of colour by interposition between a polarizer and an analyser, is by far the most delicate test of double refraction. Many organic bodies (for example, grains of starch) are thus found to be doubly refracting; and microscopists often avail themselves of this means of detecting diversities of structure in the objects which they examine.

1112. Rotation of Plane of Polarization.—When a plate of quartz (rock-crystal), even of considerable thickness, cut perpendicular to the axis, is interposed between the polarizer and analyser, colour is exhibited, the tints changing as the analyser is rotated; and similar effects of colour are produced by employing, instead of quartz, a solution of sugar, inclosed in a tube with plane glass ends.

If homogeneous light is employed, it is found that if the analyser is first adjusted to produce extinction of the polarized light, and the quartz or saccharine solution is then introduced, there is a partial restoration of light. On rotating the analyser through a certain angle, there is again complete extinction of the light; and on comparing different plates of quartz, it will be found that the angle through which the analyser must be rotated is proportional to the thickness of the plate. In the case of solutions of sugar, the angle is proportional jointly to the length of the tube and the strength of the solution.

The action thus exerted by quartz or sugar is called *rotation of the plane of polarization,* a name which precisely expresses the observed phenomena. In the case of ordinary quartz, and solutions of sugar-candy, it is necessary to rotate the analyser in the direction of watch-hands as seen by the observer, and the rotation of the plane of polarization is said to be *right-handed.* In the case of what is called *left-handed* quartz, and of solutions of non-crystallizable sugar, the rotation of the plane of polarization is in the

opposite direction, and the observer must rotate the analyser against watch-hands.

The amount of rotation is different for the different elementary colours, and has been found to be inversely as the square of the wave-length. Hence the production of colour.

1113. Magneto-optic Rotation.—Faraday made the remarkable discovery that the plane of polarization can be rotated in certain circumstances by the action of magnetism. Let a long rectangular piece of "heavy-glass" (silico-borate of lead) be placed longitudinally between the poles of the powerful electro-magnet represented in Fig. 445 (page 683), which is for this purpose made hollow in its axis, so that an observer can see through it from end to end. Let a Nicol's prism be fitted into one end of the magnet, to serve as polarizer, and another into the other end to serve as analyser, and let one of them be turned till the light is extinguished. Then, as long as no current is passed round the electro-magnet, the interposition of the heavy-glass will produce no effect; but the passing of a current while the heavy-glass is in its place between the poles, produces rotation of the plane of polarization in the same direction as that in which the current circulates. The amount of rotation is directly as the strength of current, and directly as the length of heavy-glass traversed by the light. Flint-glass gives about half the effect of heavy-glass, and all transparent solids and liquids exhibit an effect of the same kind in a more or less marked degree.

A steel magnet, if extremely powerful, may be used instead of an electro-magnet; and in all cases, to give the strongest effect, the lines of magnetic force should coincide with the direction of the transmitted ray.

Faraday regarded these phenomena as proving the direct action of magnetism upon light; but it is now more commonly believed that the direct effect of the magnetism is to put the particles of the transparent body in a peculiar state of strain, to which the observed optical effect is due.

In every case tried by Faraday, the direction of the rotation was the same as the direction in which the current circulated; but certain substances[1] have since been found which give rotation against the current. The law for the relative amounts of rotation of different colours is approximately the same as in the case of quartz.

[1] One such substance is a solution of Fe^2Cl^3 (old notation) in methylic (not methylated) alcohol.

The direction of rotation is with watch-hands as seen from one end of the arrangement, and against watch-hands as seen from the other; so that the same piece of glass, in the same circumstances, behaves like right-handed quartz to light entering it at one end, and like left-handed quartz to light entering it at the other.

The rotatory power of quartz and sugar appears to depend upon a certain unsymmetrical arrangement of their molecules, an arrangement somewhat analogous to the thread of a screw; right-handed and left-handed screws representing the two opposite rotatory powers. It is worthy of note that the two kinds of quartz crystallize in different forms, each of which is unsymmetrical, one being like the image of the other as seen in a looking-glass. Pasteur has conducted extremely interesting researches into the relations existing between substances which, while in other respects identical or nearly identical, differ as regards their power of producing rotation. For the results we must refer to treatises on chemistry.

Dr. Kerr has recently obtained rotation of the plane of polarization by reflection from intensely magnetized iron. In some of the experiments the direction of magnetization was normal, and in others parallel to the reflecting surface.

1114. Circular Polarization. Fresnel's Rhomb.—We have explained in § 1108 the process by which elliptic polarization is brought about, when plane-polarized light is transmitted through a thin plate of selenite. To obtain circular polarization (which is merely a case of elliptic), the plate must be of such thickness as to retard one component more than the other by a *quarter of a wave-length*, and must be laid so that the directions of the two component vibrations make angles of 45° with the plane of polarization. Plates specially prepared for this purpose are in general use, and are called *quarter-wave plates*. They are usually of mica, which differs but little in its properties from selenite. It is impossible, however, in this way to obtain complete circular polarization of ordinary white light, since different thicknesses are required for light of different wave-lengths, the thickness which is appropriate for violet being too small for red.

Fresnel discovered that plane-polarized light is elliptically polarized by *total internal reflection* in glass, whenever the plane of polarization of the incident light is inclined to the plane of incidence. The rectilinear vibrations of the incident light are in fact resolved into two components, one of them in, and the other per-

pendicular to, the plane of incidence; and one of these is retarded with respect to the other in the act of reflection, by an amount depending on the angle of incidence. He determined the magnitude of this angle for which the retardation is precisely $\frac{1}{8}$ of a wave-length; and constructed a *rhomb*, or oblique parallelepiped of glass (Fig. 782), in which a ray, entering normally at one end, undergoes two successive reflections at this angle (about 55°), the plane of reflection being the same in both. The total retardation of one component on the other is thus $\frac{1}{4}$ of a wave-length; and if the rhomb is in such a position that the plane in which the two reflections take place is at an angle of 45° to the plane of polarization of the incident light, the emergent light is circularly polarized. The effect does not vary much with the wave-length, and sensibly white circularly polarized light can accordingly be obtained by this method.

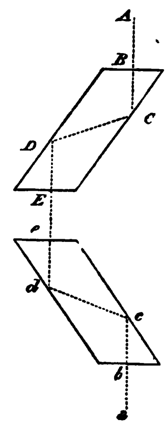


Fig. 782.
Two Fresnel's Rhombs.

When circularly polarized light is transmitted through a Fresnel's rhomb, or through a quarter-wave plate, it becomes plane-polarized, and we have thus a simple mode of distinguishing circularly polarized light from common light; for the latter does not become polarized when thus treated. Two quarter-wave plates, or two Fresnel's rhombs, may be combined either so as to assist or to oppose one another. By the former arrangement, which is represented in Fig. 782, we can convert plane-polarized light into light polarized in a perpendicular plane, the final result being therefore the same as if the plane of polarization had been rotated through 90°. The several steps of the process are illustrated by the five diagrams of Fig. 783,

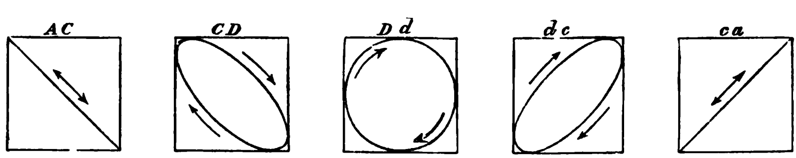


Fig. 783.—Form of Vibration in traversing the Rhombs.

which represent the vibrations of the five portions AC, CD, D*d*, *dc*,*ca* of the ray which traverses the two rhombs in the preceding figure. The sides of the square are parallel to the directions of resolution; the initial direction of vibration is one diagonal of the square, and

the final direction is the other diagonal; a gain or loss of half a complete vibration on the part of either component being just sufficient to effect this change.

1115. Direction of Vibration of Plane-polarized Light.—The plane of polarization of plane-polarized light may be defined as the plane in which it is most copiously reflected. It is perpendicular to the plane in which the light refuses to be reflected (at the polarizing angle); and is identical with the original plane of reflection, if the polarization was produced by reflection. This definition is somewhat arbitrary, but has been adopted by universal consent.

When light is polarized by the double refraction of Iceland-spar, or of any other uniaxal crystal, it is found that the plane of polarization of the ordinary ray is the plane which contains the axis of the crystal. But the distinctive properties of the ordinary ray are most naturally explained by supposing that its vibrations are perpendicular to the axis. Hence we conclude that the direction of vibration in plane-polarized light is normal to the so-called plane of polarization, and therefore that, in polarization by reflection, the vibrations of the reflected light are parallel to the reflecting surface.

This is Fresnel's doctrine. MacCullagh, however, reversed this hypothesis, and maintained that the direction of vibration is *in* the plane of polarization. Both theories have been ably expounded; but Stokes contrived a crucial experiment in diffraction, which confirmed Fresnel's view;[1] and in his classical paper on "Change of Refrangibility," he has deduced the same conclusion from a consideration of the phenomena of the polarization of light by reflection from excessively fine particles of solid matter in suspension in a liquid.[2]

1116. Vibrations of Ordinary Light.—Ordinary light agrees with circularly polarized light in always yielding two beams of equal intensity when subjected to double refraction; but it differs from circularly polarized light in not becoming plane-polarized by transmission through a Fresnel's rhomb or a quarter-wave plate. What, then, can be the form of vibration for common light? It is probably very irregular, consisting of ellipses of various sizes, positions, and forms (including circles and straight lines), rapidly succeeding one another. By this irregularity we can account for the fact that beams of light from different sources (even from different points of the same flame, or from different parts of the sun's disc), cannot, by

[1] *Cambridge Transactions.* 1850.
[2] *Philosophical Transactions,* 1852; pp. 530, 531.

any treatment whatever, be made to exhibit the phenomena of mutual interference; and for the additional fact that the two rectangular components into which a beam of common light is resolved by double refraction, cannot be made to interfere, even if their planes of polarization are brought into coincidence by one of the methods of rotation above described.

Certain phenomena of interference show that a few hundred consecutive vibrations of common light may be regarded as similar; but as the number of vibrations in a second is about 500 millions of millions, there is ample room for excessive diversity during the time that one impression remains upon the retina.

1117. Polarization of Radiant Heat.—The fundamental identity of radiant heat and light is confirmed by thermal experiments on polarization. Such experiments were first successfully performed by Forbes in 1834, shortly after Melloni's invention of the thermo-multiplier. He first proved the polarization of heat by tourmaline; next by transmission through a bundle of very thin mica plates, inclined to the transmitted rays; and afterwards by reflection from the multiplied surfaces of a pile of thin mica plates placed at the polarizing angle. He next succeeded in showing that polarized heat, even when quite obscure, is subject to the same modifications which doubly refracting crystallized bodies impress upon light, by suffering a beam of heat, after being polarized by transmission, to pass through an interposed plate of mica, serving the purpose of the plate of selenite in the experiment of § 1107, the heat traversing a second mica bundle before it was received on the thermo-pile. As the interposed plate was turned round in its own plane, the amount of heat shown by the galvanometer was found to fluctuate just as the amount of light received by the eye under similar circumstances would have done. He also succeeded in producing circular polarization of heat by a Fresnel's rhomb of rock-salt. These results have since been fully confirmed by the experiments of other observers.

EXAMPLES IN ACOUSTICS.

Period, Wave-length, and Velocity.

1. If an undulation travels at the rate of 100 ft. per second, and the wave-length is 2 ft., find the period of vibration of a particle, and the number of vibrations which a particle makes per second.

2. It is observed that waves pass a given point once in every 5 seconds, and that the distance from crest to crest is 20 ft. Find the velocity of the waves in feet per second.

3. The lowest and highest notes of the normal human voice have about 80 and 800 vibrations respectively per second. Find their wave-lengths when the velocity of sound is 1100 ft. per second.

4. Find their wave-lengths in water in which the velocity of sound is 4900 feet per second.

5. Find the wave-length of a note of 500 vibrations per second in steel in which the velocity of propagation is 15,000 ft. per second.

Pitch and Musical Intervals.

6. Show that a "fifth" added to a "fourth" makes an octave.

7. Calling the successive notes of the gamut Do_1, Re_1, Mi_1, Fa_1, Sol_1, La_1, Si_1, Do_2, show that the interval from Sol_1 to Re_2 is a true "fifth."

8. Find the first 5 harmonics of Do_1.

9. A siren of 15 holes makes 2188 revolutions in a minute when in unison with a certain tuning-fork. Find the number of vibrations per second made by the fork.

10. A siren of 15 holes makes 440 revolutions in a quarter of a minute when in unison with a certain pipe. Find the note of the pipe (in vibrations per second).

Reflection of Sound, and Tones of Pipes.

11. Find the distance of an obstacle which sends back the echo of a sound to the source in $1\frac{1}{2}$ seconds, when the velocity of sound is 1100 ft. per second.

12. A well is 210 ft. deep to the surface of the water. What time will elapse between producing a sound at its mouth and hearing the echo?

13. What is the wave-length of the fundamental note of an open organ-pipe 16 ft. long; and what are the wave-lengths of its first two overtones? Find also their vibration-numbers per second.

14. What is the wave-length of the fundamental tone of a stopped organ pipe

5 ft. long; and what are the wave-lengths of its first two overtones? Find also their vibration-numbers per second.

15. What should be the length of a tube stopped at one end that it may resound to the note of a tuning-fork which makes 520 vibrations per second; and what should be the length of a tube open at both ends that it may resound to the same fork. [The tubes are supposed narrow, and the smallest length that will suffice is intended.]

16. Would tubes twice as long as those found in last question resound to the fork? Would tubes three times as long?

Beats.

17. One fork makes 256 vibrations per second, and another makes 260. How many beats will they give in a second when sounding together?

18. Two sounds, each consisting of a fundamental tone with its first two harmonics, reach the ear together. One of the fundamental tones has 300 and the other 302 vibrations per second. How many beats per second are due to the fundamental tones, how many to the first harmonics, and how many to the second harmonics?

19. A note of 225 vibrations per second, and another of 336 vibrations per second, are sounded together. Each of the two notes contains the first two harmonics of the fundamental. Show that two of the harmonics will yield beats at the rate of 3 per second.

Velocity of Sound in Gases.

20. If the velocity of sound in air at 0° C. is 33,240 cm. per second, find its velocity in air at 10° C., and in air at 100° C.

21. If the velocity of sound in air at 0° C. is 1090 ft. per second, what is the velocity in air at 10°?

22. Show that the difference of velocity for 1° of difference of temperature in the Fahrenheit scale is about 1 ft. per second.

23. If the wave-length of a certain note be 1 metre in air at 0°, what is it in air at 10°?

24. The density of hydrogen being ·06926 of that of air at the same pressure and temperature, find the velocity of sound in hydrogen at a temperature at which the velocity in air is 1100 ft. per second.

25. The quotient of pressure (in dynes per sq. cm.) by density (in gm. per cubic cm.) for nitrogen at 0° C. is 807 million. Compute (in cm. per second) the velocity of sound in nitrogen at this temperature.

26. If a pipe gives a note of 512 vibrations per second in air, what note will it give in hydrogen?

27. A pipe gives a note of 100 vibrations per second at the temperature 10° C. What must be the temperature of the air that the same pipe may yield a note higher by a major fifth?

Vibrations of Strings.

28. Find, in cm. per second, the velocity with which pulses travel along a string whose mass per cm. of length is ·005 gm., when stretched with a force of 7 million dynes.

29. If the length of the string in last question be 33 cm., find the number of vibrations that it makes per second when vibrating in its fundamental mode; also the numbers corresponding to its first two overtones.

30. The A string of a violin is 33 cm. long, has a mass of ·0065 gm. per cm., and makes 440 vibrations per second. Find the stretching force in dynes.

31. The E string of a violin is 33 cm. long, has a mass of ·004 gm. per cm., and makes 660 vibrations per second. Find the stretching force in dynes.

32. Two strings of the same length and section are formed of materials whose specific gravities are respectively d and d'. Each of these strings is stretched with a weight equal to 1000 times its own weight. What is the musical interval between the notes which they will yield?

33. The specific gravity of platinum being taken as 22, and that of iron as 7·8, what must be the ratio of the lengths of two wires, one of platinum and the other of iron, both of the same section, that they may vibrate in unison when stretched with equal forces?

Longitudinal Vibrations of Rods.

34. If sound travels along fir in the direction of the fibres at the rate of 15,000 ft. per second, what must be the length of a fir rod that, when vibrating longitudinally in its fundamental mode, it may emit a note of 750 vibrations per second?

35. A rod 8 ft. long, vibrating longitudinally in its fundamental mode, gives a note of 800 vibrations per second. Find the velocity with which pulses are propagated along it.

EXAMPLES IN OPTICS.

Photometry, Shadows, and Plane Mirrors.

36. A lamp and a taper are at a distance of 4·15 m. from each other; and it is known that their illuminating powers are as 6 to 1. At what distance from the lamp, in the straight line joining the flames, must a screen be placed that it may be equally illuminated by them both?

37. Two parallel plane mirrors face each other at a distance of 3 ft., and a small object is placed between them at a distance of 1 ft. from the first mirror, and therefore of 2 ft. from the second. Calculate the distances, from the first mirror, of the three nearest images which are seen in it; and make a similar calculation for the second mirror.

38. Show that a person standing upright in front of a vertical plane mirror will just be able to see his feet in it, if the top of the mirror is on a level with his eyes, and its height from top to bottom is half the height of his eyes above his feet.

39. A square plane mirror hangs exactly in the centre of one of the walls of a cubical room. What must be the size of the mirror that an observer with his

eye exactly in the centre of the room may just be able to see the whole of the opposite wall reflected in it except the part concealed by his body?

40. Two plane mirrors contain an angle of 160°, and form images of a small object between them. Show that if the object be within 20° of either mirror there will be three images; and that if it be more than 20° from both, there will be only two.

41. Show that when the sun is shining obliquely on a plane mirror, an object directly in front of the mirror may give two shadows, besides the direct shadow.

42. A person standing beside a river near a bridge observes that the inverted image of the concavity of the arch receives his shadow exactly as a real inverted arch would do if it were in the place where the image appears to be. Explain this.

43. If a globe be placed upon a table, show that the breadth of the elliptic shadow cast by a candle (considered as a luminous point) will be independent of the position of the globe.

44. What is the length of the cone of the *umbra* thrown by the earth? and what is the diameter of a cross section of it made at a distance equal to that of the moon?

The radius of the sun is 112 radii of the earth; the distance of the moon from the earth is 60 radii of the earth; and the distance of the sun from the earth is 24,000 radii of the earth. Atmospheric refraction is to be neglected.

45. The stem of a siren carries a plane mirror, thin, polished on both sides, and parallel to the axis of the stem. The siren gives a note of 345 vibrations per second. The revolving plate has 15 holes. A fixed source of light sends to the mirror a horizontal pencil of parallel rays. What space is traversed in a second by a point of the reflected pencil at a distance of 4 metres from the axis of the siren? This axis is supposed vertical.

Spherical Mirrors.

46. Find the focal length of a concave mirror whose radius of curvature is 2 ft., and find the position of the image (*a*) of a point 15 in. in front of the mirror; (*b*) of a point 10 ft. in front of the mirror; (*c*) of a point 9 in. in front of the mirror; (*d*) of a point 1 in. in front of the mirror.

47. Calling the diameter of the object unity, find the diameters of the image in the four preceding cases.

48. The flame of a candle is placed on the axis of a concave spherical mirror at the distance of 154 cm., and its image is formed at the distance of 45 cm. What is the radius of curvature of the mirror?

49. On the axis of a concave spherical mirror of 1 m. radius, an object 9 cm. high is placed at a distance of 2 m. Find the size and position of the image.

50. What is the size of the circular image of the sun which is formed at the principal focus of a mirror of 20 m. radius? The apparent diameter of the sun is 30′.

51. In front of a concave spherical mirror of 2 metres' radius is placed a concave luminous arrow, 1 decimetre long, perpendicular to the principal axis, and at the distance of 5 metres from the mirror. What are the position and size of the image? A small plane reflector is then placed at the principal focus of the spherical mirror, at an inclination of 45° to the principal axis, its polished side being next the mirror. What will be the new position of the image?

Refraction.

(The index of refraction of glass is to be taken as $\frac{3}{2}$, except where otherwise specified, and the index of refraction of water as $\frac{4}{3}$).

52. The sine of 45° is $\sqrt{\frac{1}{2}}$, or ·707 nearly. Hence, determine whether a ray incident in water at an angle of 45° with the surface will emerge or will be reflected; and determine the same question for a ray in glass.

53. If the index of refraction from air into crown-glass be $1\frac{1}{2}$, and from air into flint-glass $1\frac{2}{3}$, find the index of refraction from crown-glass into flint-glass.

54. The index of refraction from water into oil of turpentine is 1·11; find the index of refraction from air into oil of turpentine.

55. The index of refraction for a certain glass prism is 1·6, and the angle of the prism is 10°. Find approximately the deviation of a ray refracted through it nearly symmetrically.

56. A ray of light falls perpendicularly on the surface of an equilateral prism of glass with a refracting angle of 60°. What will be the deviation produced by the prism? Index of refraction of glass 1·5.

57. A speck in the interior of a piece of plate-glass appears to an observer looking normally into the glass to be 2 mm. from the near surface. What is its real distance?

58. The rays of a vertical sun are brought to a focus by a lens at a distance of 1 ft. from the lens. If the lens is held just above a smooth and deep pool of water, at what depth in the water will the rays come to a focus?

59. A mass of glass is bounded by a convex surface, and parallel rays incident nearly normally on this surface come to a focus in the interior of the glass at a distance a. Find the focal length of a plano-convex lens of the same convexity, supposing the rays to be incident on the convex side.

60. Show that the deviation of a ray going through an air-prism in water is towards the edge of the prism.

Lenses, &c.

61. Compare the focal lengths of two lenses of the same size and shape, one of glass and the other of diamond, their indices of refraction being respectively 1·6 and 2·6.

62. If the index of refraction of glass be $\frac{3}{2}$, show that the focal length of an equi-convex glass lens is the same as the radius of curvature of either face.

63. The focal length of a convex lens is 1 ft. Find the positions of the image of a small object when the distances of the object from the lens are respectively 20 ft., 2 ft., and $1\frac{1}{2}$ ft. Are the images real or virtual?

64. When the distances of the object from the lens in last question are respectively 11 in., 10 in., and 1 in., find the distances of the image. Are the images real or virtual?

65. Calling the diameter of the object unity, find the diameter of the image in the six cases of questions 63, 64, taken in order.

66. Show that, when the distance of an object from a convex lens is double the focal length, the image is at the same distance on the other side.

67. The object is 6 ft. on one side of a lens, and the image is 1 ft. on the other side. What is the focal length of the lens?

68. The object is 3 in. from a lens, and its image is 18 in. from the lens on the same side. Is the lens convex or concave, and what is its focal length?

69. The object is 12 ft. from a lens, and the image 1 ft. from the lens on the same side. Find the focal length, and determine whether the lens is convex or concave.

70. A person who sees best at the distance of 3 ft., employs convex spectacles with a focal length of 1 ft. At what distance should he hold a book, to read it with the aid of these spectacles?

71. A person reads a book at the distance of 1 ft. with the aid of concave spectacles of 6 in. focal length. At what distance is the image which he sees?

72. A pencil of parallel rays fall upon a sphere of glass of 1 inch radius. Find the principal focus of rays near the axis, the index of refraction of glass being 1·5.

73. What is the focal length of a double-convex lens of diamond, the radius of curvature of each of its faces being 4 millimetres? Index of refraction 2·5.

74. An object 8 centimetres high is placed at 1 metre distance on the axis of an equi-convex lens of crown-glass of index 1·5, the radius of curvature of its faces being 0·4 m. Find the size and position of the image.

75. Two converging lenses, with a common focal length of 0·05 m., are at a distance of 0·05 m. apart, and their axes coincide. What image will this system give of a circle 0·01 m. in diameter, placed at a distance of ·1 m. on the prolongation of the common axis?

76. Show that if F denote the focal length of a combination of two lenses in contact, their thicknesses being neglected, we have

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2},$$

f_1 and f_2 denoting the focal lengths of the two lenses.

77. What is the focal length of a lens composed of a convex lens of 2 in. focal length, cemented to a concave lens of 9 in. focal length?

78. Apply the formulæ of § 1015 to find the focal length of a lens, the thickness being neglected.

79. The objective of a telescope has a focal length of 20 ft. What will be the magnifying power when an eye-piece of half-inch focus is used?

80. A sphere of glass of index 1·5 lying upon a horizontal plane receives the sun's rays. What must be the height of the sun above the horizon that the principal focus of the sphere may be in this horizontal plane?

81. A small plane mirror is placed exactly at the principal focus of a telescope, nearly perpendicular to its axis, and the telescope is directed approximately to a distant luminous object. Show that the rays reflected at the mirror will, after repassing the object glass, return in the exact direction from which they came, in spite of the small errors of adjustment of the mirror and telescope.

82. An eye is placed close to the surface of a large sphere of glass ($\mu = \frac{3}{2}$) which is silvered at the back. Show that the image which the eye sees of itself is three-fifths of the natural size.

83. The refractive indices for the rays D and F for two specimens of glass are

Crown-glass	......	1·5279	......	1·5344
Flint-glass	......	1·6351	......	1·6481

and an achromatic lens of 20 in. focal length is to be formed by their combination. Show that if the rays D and F are brought to the same focus, the sum of curvatures of the two faces for the crown lens must be double that for the flint, and the focal lengths of the two lenses which are combined will be about 7·9 in. for the crown and 13·1 in. for the flint.

ANSWERS TO EXAMPLES IN ACOUSTICS.

Ex. 1. $\frac{1}{50}$ sec. 50. Ex. 2. 4. Ex. 3. $13\frac{3}{4}$ ft. $1\frac{3}{8}$ ft. Ex. 4. $61\frac{1}{4}$ ft., $6\frac{1}{8}$ ft. Ex. 5. 30 ft.

Ex. 6. $\frac{3}{2} \times \frac{4}{3} = 2$. Ex. 7. $\frac{36}{54} = \frac{2}{3}$. Ex. 8. Do_2, Sol_2, Do_3, Mi_3, Sol_3.

Ex. 9. 547. Ex. 10. 440.

Ex. 11. 825. Ex. 12. $\frac{21}{55} = {\cdot}382$ sec. Ex. 13. 32 ft., 16 ft., $10\frac{2}{3}$ ft.; $34\frac{3}{8}$, $68\frac{3}{4}$, $103\frac{1}{8}$. Ex. 14. 20 ft., $6\frac{2}{3}$ ft., 4 ft.; 55, 165, 275. Ex. 15. $\frac{55}{104}$ ft., $\frac{55}{52}$ ft. Ex. 16. An open tube twice or three times as long will resound, because one of its overtones will coincide with the note of the fork. A stopped tube three times as long will resound, but a stopped tube twice as long will not.

Ex. 17. 4. Ex. 18. 2, 4, 6. Ex. 19. $675 - 672 = 3$.

Ex. 20. 33843, 38850. Ex. 21. 1110. Ex. 22. The velocity is 1090 at 32° and 1110 at 50°. Ex. 23. 1·018 metre. Ex. 24. 4180 ft. per second. Ex. 25. 33732. Ex. 26. 1945 vibrations per second. Ex. 27. 364° C.

Ex. 28. 37417. Ex. 29. 567, 1134, 1701. Ex. 30. $v = 29040$, $t = v^2 m = 5481600$. Ex. 31. $v = 43560$, $t = 7589900$. Ex. 32. Unison. Ex. 33. Length of iron = 1·68 times length of platinum.

Ex. 34. 10 ft. Ex. 35. 12800 ft. per second.

ANSWERS TO EXAMPLES IN OPTICS.

Ex. 36. 2·95 m. Ex. 37. 1, 5, and 7 ft. behind first mirror; 2, 4, and 8 ft. behind second. Ex. 39. Side of mirror must be $\frac{1}{3}$ of edge of cube.

Ex. 41. They are the shadows of the object and of its image, cast by the sun's image. The former is due to the intercepting of light after reflection; the latter to the intercepting of light before reflection. Ex. 42. The sun's image throws a shadow of the man's image on the real arch, owing to his intercepting rays on their way to the water. Ex. 43. First let the globe be vertically under the flame, and draw through the flame two equally inclined planes, touching the globe. Their intersections with the table will be parallel lines which will be tangents to the shadow, and will still remain tangents to it as the globe is rolled between the planes to any distance. Ex. 44. 216 radii of earth; $1\frac{4}{9}$ radii. Ex. 45. $368\pi = 1156$ metres.

Ex. 46. Focal length 1 ft.; (*a*) 5 ft. in front of mirror; (*b*) $1\frac{1}{9}$ ft. in front; (*c*) 3 ft. behind mirror; (*d*) $1\frac{1}{11}$ in. behind. Ex. 47. 4, $\frac{1}{9}$, 4, $1\frac{1}{11}$.

Ex. 48. 69·6 cm. Ex. 49. Distance $\frac{2}{3}$ m., height 3 cm. Ex. 50. 8·73 cm. Ex. 51. Distance $1\frac{1}{4}$ m., length $\frac{1}{4}$ dec., new position $\frac{1}{4}$ m. laterally from focus.

Ex. 52. The ray in water will emerge, because $\frac{3}{4}$ is greater than ·707; the ray

in glass will be totally reflected, because $\frac{2}{3}$ is less than ·707. Ex. 53. $\frac{16}{15}$. Ex. 54. 1·48. Ex. 55. 6°. Ex. 56. 60° (by total reflection).

Ex. 57. 3 mm. Ex. 58. 1 ft. 4 in. Ex. 59. $\frac{2}{3}\,a$.

Ex. 61. Focal length of diamond lens is $\frac{3}{8}$ of focal length of glass lens. Ex. 63. $1\frac{1}{19}$ ft., 2 ft., 3 ft. on other side of lens. All real. Ex. 64. 11 ft., 5 ft., $\frac{1}{11}$ ft. on same side of lens. All virtual. Ex. 65. $\frac{1}{19}$, 1, 2, 12, 6, $1\frac{1}{11}$. Ex. 66. $1\frac{1}{11}$. Ex. 67. $\frac{6}{7}$ ft. Ex. 68. $3\frac{3}{5}$ in., convex. Ex. 69. $1\frac{1}{11}$ ft., concave. Ex. 70. 9 in. Ex. 71. 4 in. Ex. 72. 1·5 in. from centre, or ·5 in. from sphere.

Ex. 73. $1\frac{1}{3}$ mm. Ex. 74. Distance $\frac{2}{3}$ m. on other side, height $5\frac{1}{3}$ cm. Ex. 75. A real image ·025 m. beyond second lens; diameter of image ·005 m. Ex. 77. $2\frac{4}{7}$ in. Ex. 79. 480. Ex. 80. Sine of altitude $= \frac{2}{3}$, altitude $= 41° \, 49'$. Ex. 81. Rays from one point of object converge to one point on mirror, and are reflected from this point as a new source. Hence by the principle of conjugate foci they will return to the point whence they came. Ex. 82. The first and second images are at distances of $\frac{1}{3}$ and $\frac{2}{3}$ of radius from centre.

Ex. 83. The dispersive powers are as 32 : 53. The focal lengths are to be directly as these numbers, and the difference of their reciprocals must be $\frac{1}{50}$.

INDEX TO PART IV.

THE END.